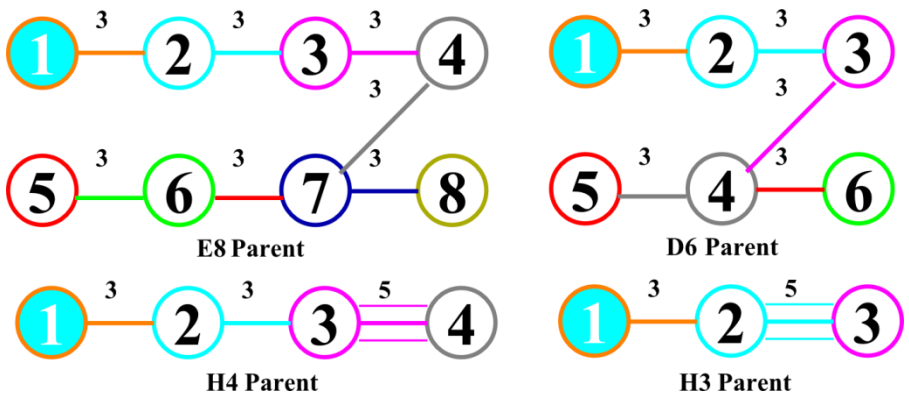


3D & 4D Solids using Quaternion Weyl Orbits from Coxeter-Dynkin Geometric Group Theory

0D points & vertices

1D lines & edges

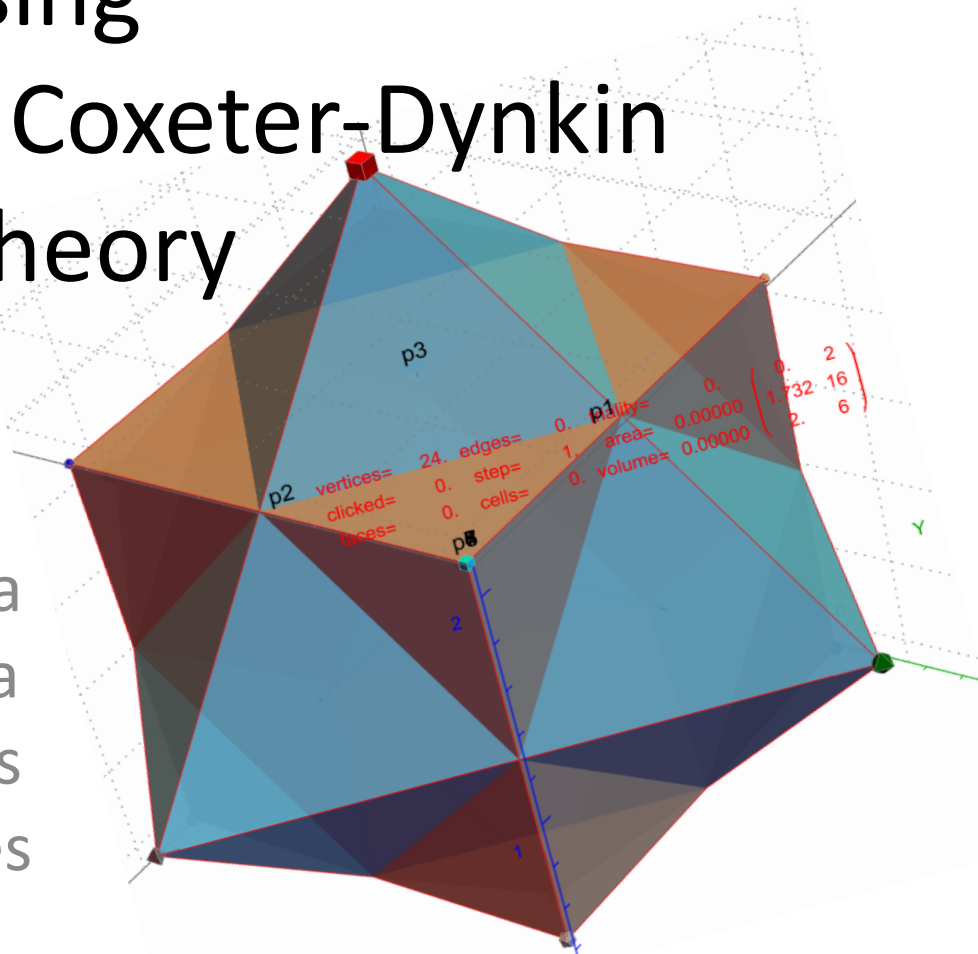
2D faces & planes

3D cells & polyhedra

4D cells & polychora

5D-8D cells & polytopes

∞D cells & polytopes



Agenda

- Introducing the Mathematica™ based VisibLie-E8 Tool
  - The Coxeter-Dynkin Visualizer (Groups, Weyl Orbits, Hasse Diagrams, etc.)
  - The (Bi)Quaternion/(Bi)Octonion/Sedenion Engine (Fano Plane/Cube/Tesseract, etc.)
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  - 4D cells & polychora (A4, C4, D4, F4, H4 Groups)
  - 5D-8D cells & polytopes (A5, C6, D6, E6, E7, E8=BC8+D8 Groups)
- Generating Solid Convex Hulls from Quaternions and their Weyl Group Orbits
  - The 5 Platonic Solids, which includes duals
  - The 13 Archimedean Solids and their Catalan Duals, with some Johnson and Near-miss Johnsons
- 4D Polychora w/Duals, including the Dual Snub 24-Cell

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Introducing the Mathematica™ based VisibLie-E8 Tool

J Gregory Moxness
JGMoxness@TheoryOfEverything.org

For more detail information online please click here.

Citations to source material found in this presentation:

- *Koca, Mehmet; Ozdes Koca, Nazife; Koc, Ramazon (2010). "Catalan Solids Derived From 3D-Root Systems and Quaternions". Journal of Mathematical Physics.* ***51*** *(4). arXiv:0908.3272. doi:10.1063/1.3356985.*

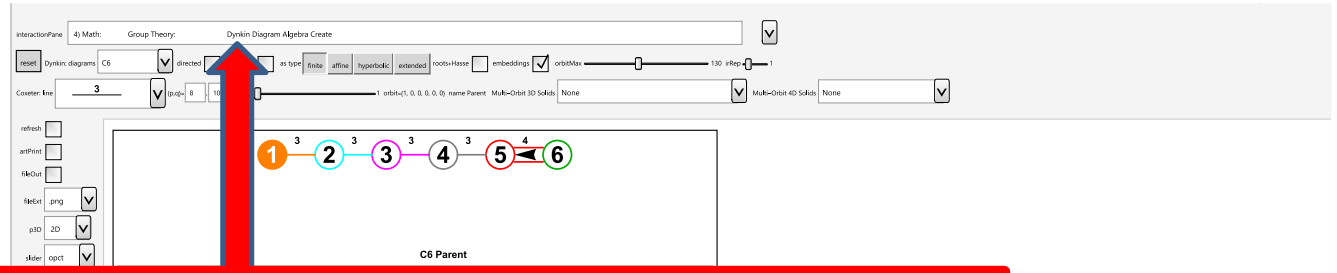
- *Koca, Mehmet; Al-Ajmi, Mudhahir; Ozdes Koca, Nazife (2011). "Quaternionic representation of snub 24-cell and its dual polytope derived from E8 root system". Linear Algebra and Its Applications.* ***434*** *(4): 977–989. doi:10.1016/j.laa.2010.10.005. ISSN 0024-3795. S2CID 18278359.*

Introducing The VisibLie_E8 Tool

VisibLie_E8

A Theory Of Everything Visualizer

J Gregory Moxness
TheoryOfEverything.org



Initializations

Analytical CodeBase

Demonstration CodeBase

- Pane Symbol Definitions
- localVisibLieE8 Manipulate
- Localized Pane Initializations (cloud mode) & UI Logic
- cloudVisibLieE8 Manipulate
- Non-localized Pane Initializations (working mode)
- Cloud Demonstrations Execution
- Non-Localized Additions
- Generate the UI

```
In[•]:= (* It takes 1 to 4 minutes to get to this point and can take 1 to 16 minutes more to display the viewer,
depending on the panes selected, the number of shells in Atoms, localizing, or using internet for curated data. *)
If[cloud,
 fixDo@cloudVisibLieE8,
 localVisibLieE8]
```

The code is written to be able to run locally with full capability or in demonstrations over the web in the cloud. It contains 18 integrated mathematical demonstration sections that work together to understand the relationships between math, physics, chemistry, genetics, neurology, AI, and more.

Just select the cells on the right and shift-click to run the entire notebook.

Introducing The VisibLie_E8 Tool

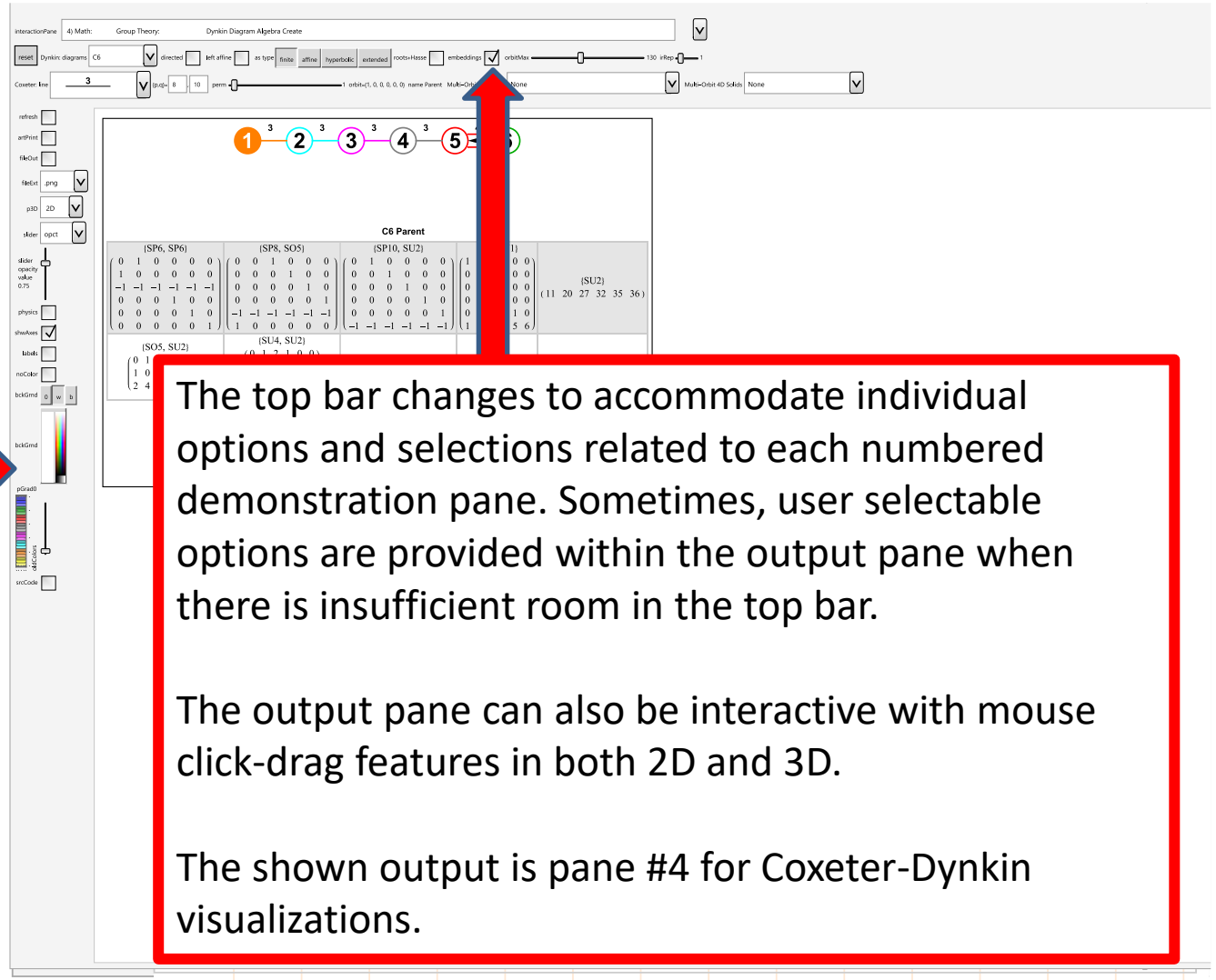
VisibLie_E8

A Theory Of Everything Visualizer

The left-side bar is common to all demonstrations and allows visualizing results in 2D, 3D, stereo and anaglyph (red-cyan glasses), control the number of time-steps for video animations, etc.

You can select color schemes, view code snippets, change opacity, resolutions, turning on/off coordinate axis, vertex labels, turn on file exports, and change export file types.

The physics button shows theoretical quantum particle assignments based on E8 group theoretic considerations.

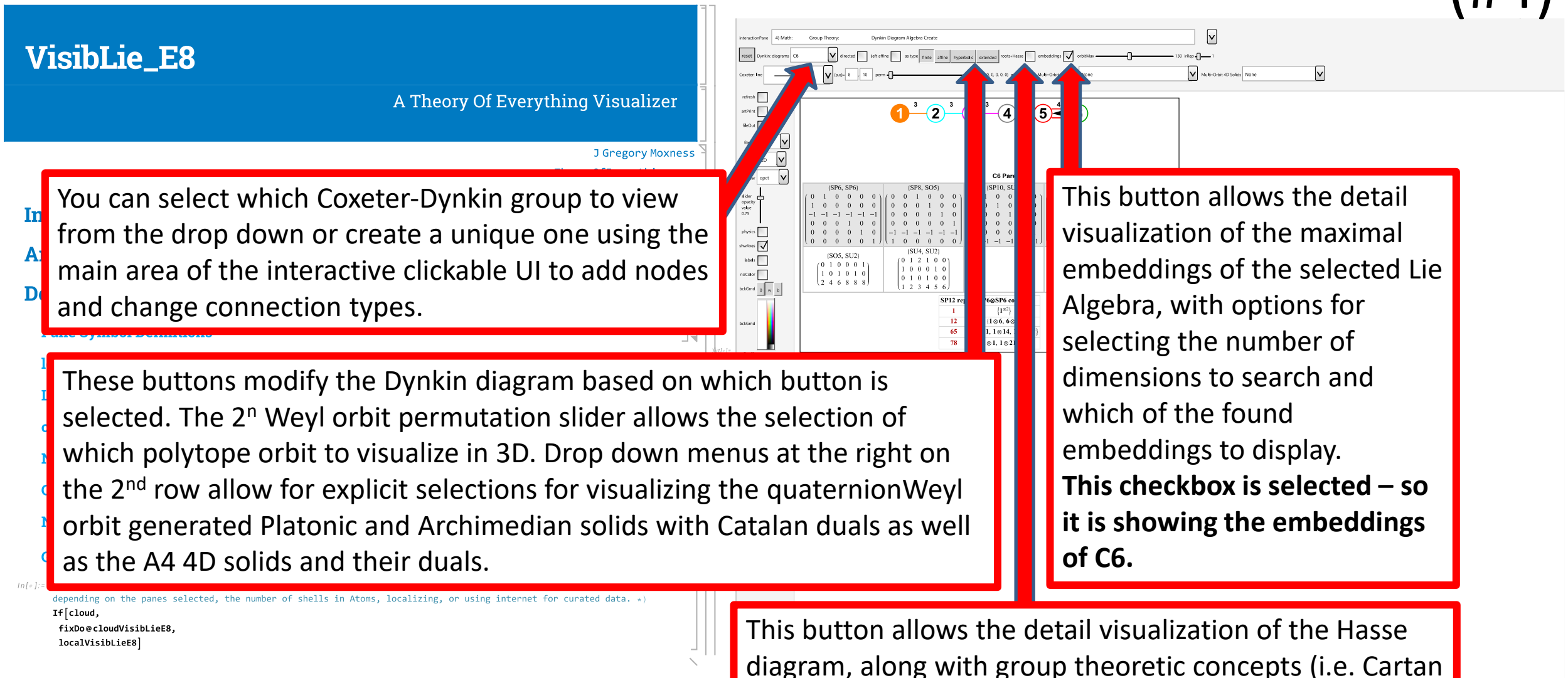


The top bar changes to accommodate individual options and selections related to each numbered demonstration pane. Sometimes, user selectable options are provided within the output pane when there is insufficient room in the top bar.

The output pane can also be interactive with mouse click-drag features in both 2D and 3D.

The shown output is pane #4 for Coxeter-Dynkin visualizations.

The Coxeter-Dynkin Visualizer Demonstration Pane (#4)



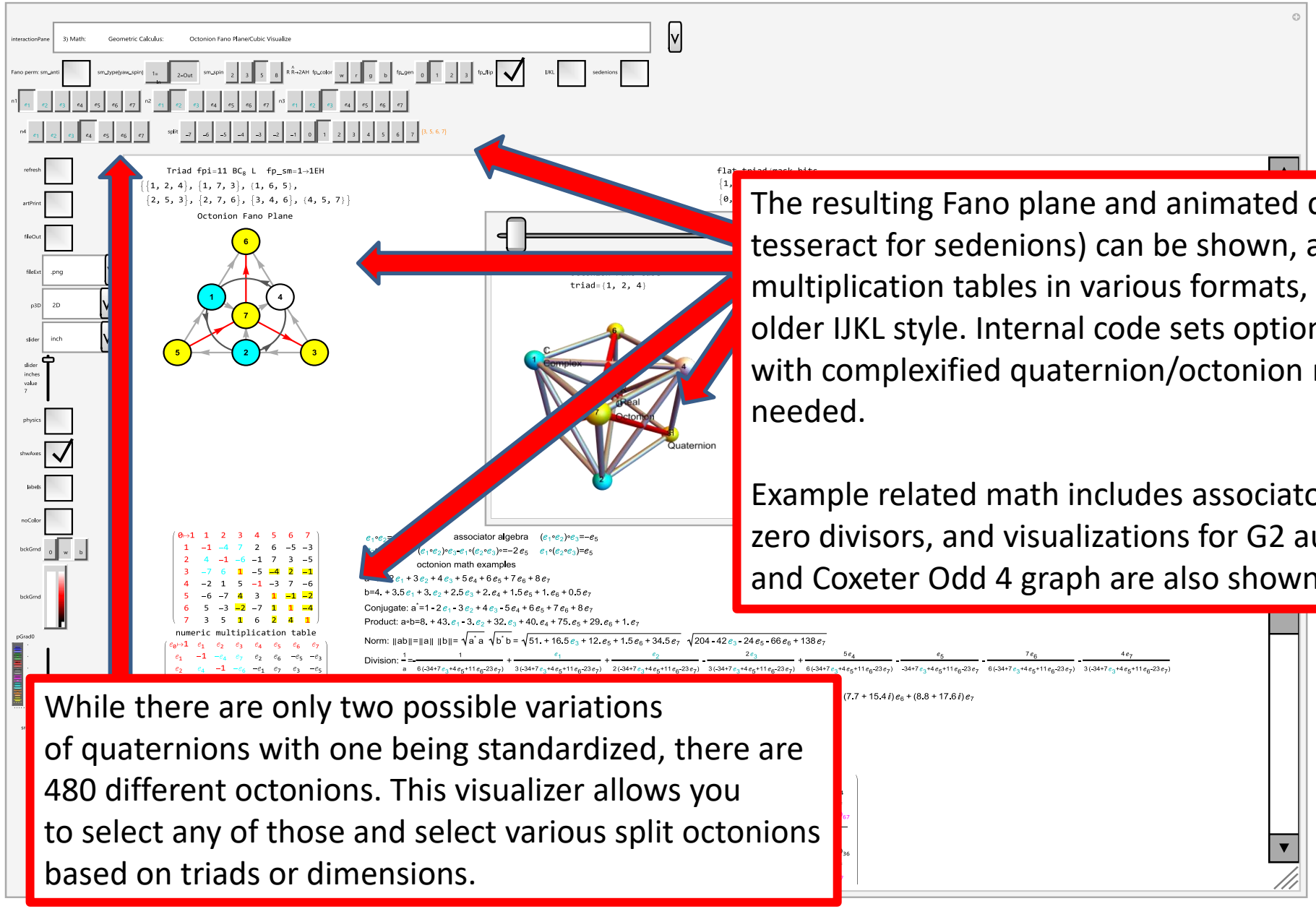
You can select which Coxeter-Dynkin group to view from the drop down or create a unique one using the main area of the interactive clickable UI to add nodes and change connection types.

These buttons modify the Dynkin diagram based on which button is selected. The 2^n Weyl orbit permutation slider allows the selection of which polytope orbit to visualize in 3D. Drop down menus at the right on the 2nd row allow for explicit selections for visualizing the quaternionWeyl orbit generated Platonic and Archimedian solids with Catalan duals as well as the A4 4D solids and their duals.

This button allows the detail visualization of the maximal embeddings of the selected Lie Algebra, with options for selecting the number of dimensions to search and which of the found embeddings to display. **This checkbox is selected – so it is showing the embeddings of C6.**

This button allows the detail visualization of the Hasse diagram, along with group theoretic concepts (i.e. Cartan matrix determinant, group dimensions, weights, positive roots, heights, etc.)

The (Bi)Quaternion / (Bi)Octonion / Sedenion Engine (#3)



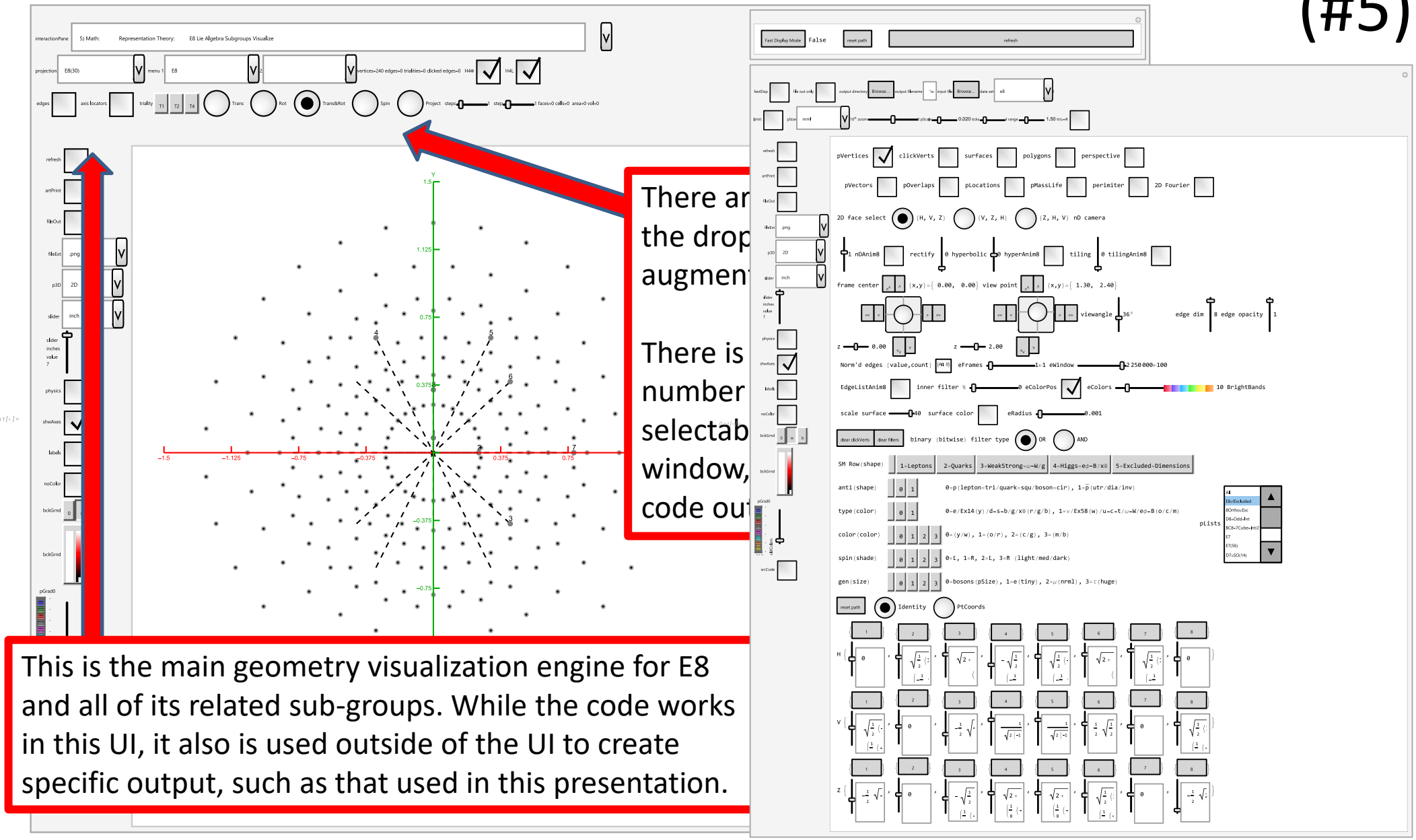
The resulting Fano plane and animated cube (or tesseract for sedenions) can be shown, along with the multiplication tables in various formats, including the older IJKL style. Internal code sets options for working with complexified quaternion/octonion math as needed.

Example related math includes associators, sedenion zero divisors, and visualizations for G2 automorphisms and Coxeter Odd 4 graph are also shown.

While there are only two possible variations of quaternions with one being standardized, there are 480 different octonions. This visualizer allows you to select any of those and select various split octonions based on triads or dimensions.

The VisibLie_E8 Hyperdimensional Visualizer Pane

(#5)



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This is the main geometry visualization engine for E8 and all of its related sub-groups. While the code works in this UI, it also is used outside of the UI to create specific output, such as that used in this presentation.

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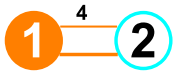
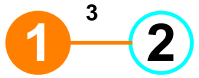
2D edges & faces (A2, B2=C2, H2, G2 Groups)



BC2 Parent

Cartan=Schlafli $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

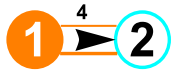
Group SU(3)



A2 Parent

Cartan $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$

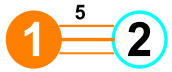
Group SO(5)



B2 Parent

Cartan $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & \frac{1}{2}\left(-1-\sqrt{5}\right) \\ \frac{1}{2}\left(-1-\sqrt{5}\right) & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$

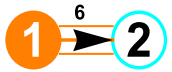
Group ⊃ SL(2,3)



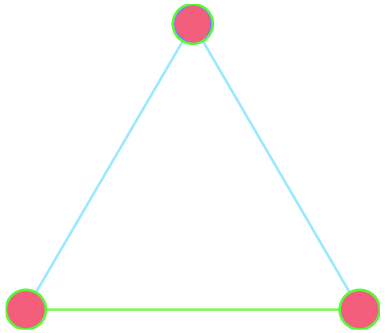
H2 Parent

Cartan $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$

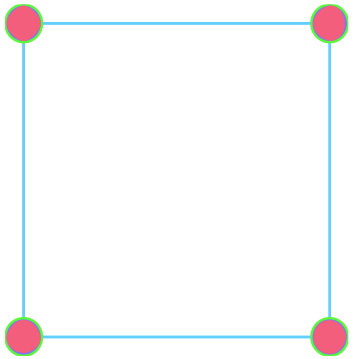
Group ⊃ O(3)



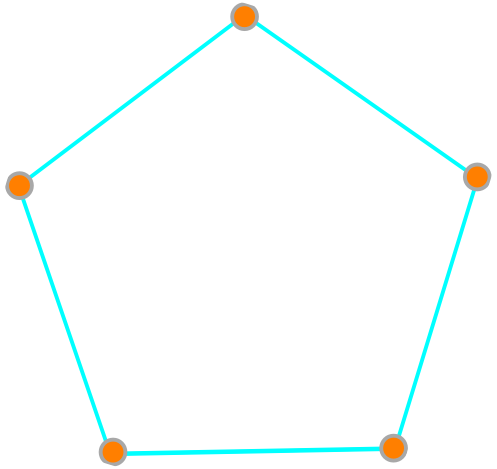
G2 Parent



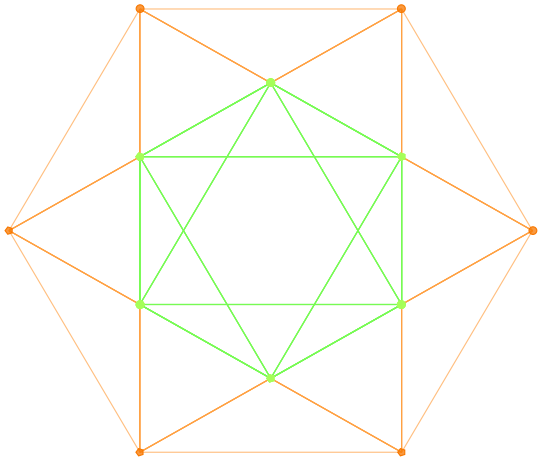
Triangle (A3)



Square (BC2)



Pentagon (H2)

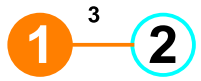


Hexagon(G2)

2D edges & faces (A2, B2=C2, H2, G2 Groups)

Cartan=Schlafli $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

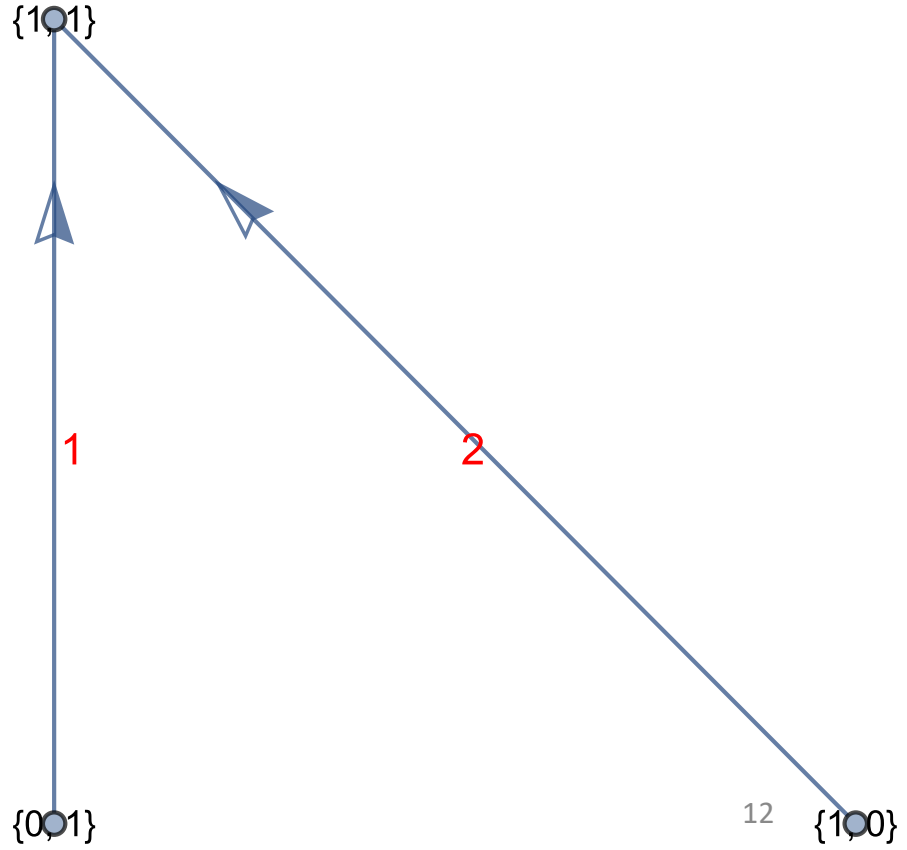
Group SU(3)



A2 Parent

Group=SU(3)

Hasse Visualization

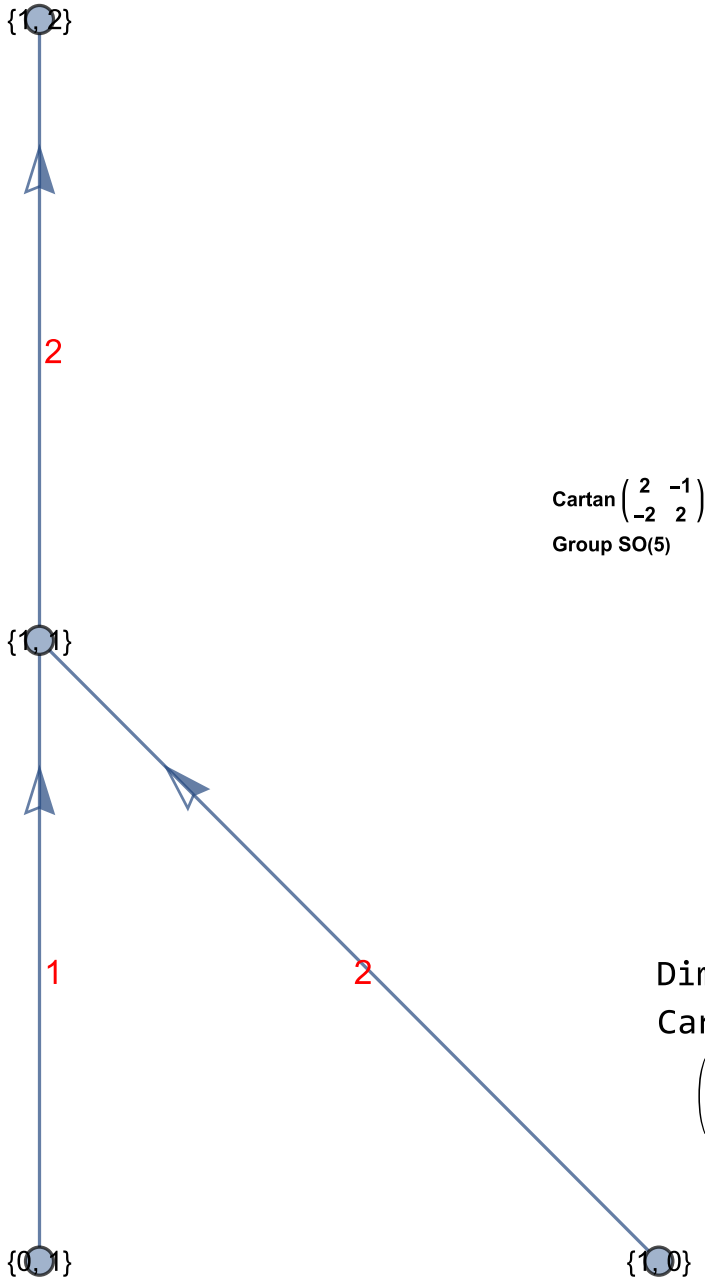


Dimension=8 Rank=2 DetCM=3 # of Positive Roots=3 Coxeter #=3

CartanMatrix	Root #	Weights	Positive Root Vectors	Heights
$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

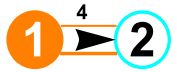
2D edges & faces (A2, B2=C2, H2, G2 Groups)

Group=SO(5)
Hasse Visualization



Cartan $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$

Group SO(5)



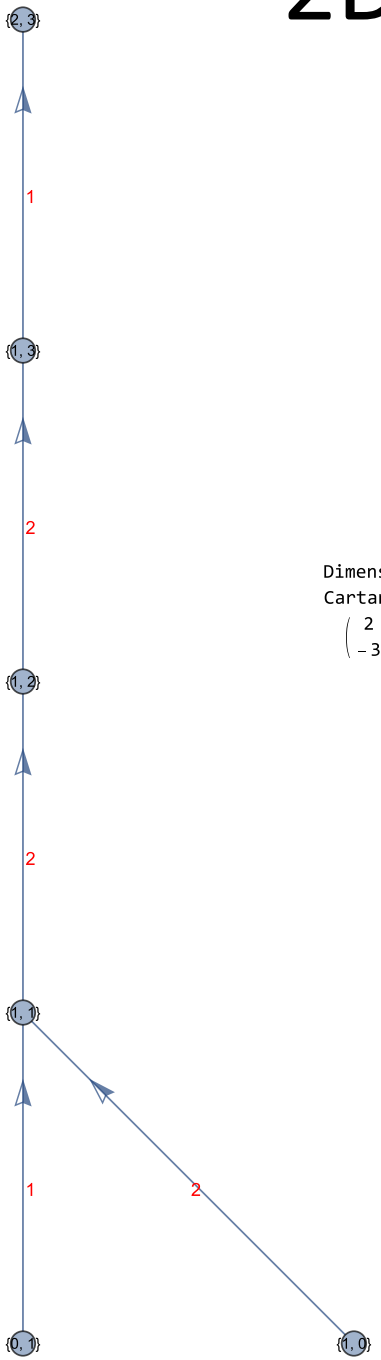
B2 Parent

Dimension=10 Rank=2 DetCM=2 # of Positive Roots=4 Coxeter #=4

CartanMatrix $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ Root # $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ Weights $\begin{pmatrix} 2 & -2 \\ -1 & 2 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ Positive Root Vectors $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ Heights $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

2D edges & faces (A2, B2=C2, H2, G2 Groups)

Group=⊃ O(3)
Hasse Visualization

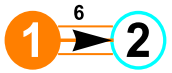


Dimension=14 Rank=2 DetCM=1 # of Positive Roots=6 Coxeter #=6

CartanMatrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ Root # $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ Weights $\begin{pmatrix} 2 & -3 \\ -1 & 2 \\ 1 & -1 \\ 0 & 1 \\ -1 & 3 \\ 1 & 0 \end{pmatrix}$ Positive Root Vectors $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}$ Heights $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$

Cartan $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ **Schlafli** $\begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}$ **Coxeter** $\begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$

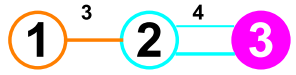
Group ⊃ O(3)



G2 Parent

3D cells & polyhedra (A3, BC3, H3)

Showing Platonic Solids



BC3 BiRectified

H3 BiRe

BC3

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Group SU(4)

$$\frac{e_1}{\sqrt{2}} + \frac{e_2}{\sqrt{2}} \quad \frac{e_3}{\sqrt{2}} - \frac{e_2}{\sqrt{2}} \quad \frac{e_2}{\sqrt{2}} - \frac{e_1}{\sqrt{2}}$$

A3 Parent

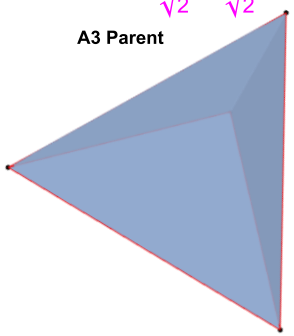
Cartan $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$ Schlafli $\begin{vmatrix} -1 & 2 & - \\ 0 & -\sqrt{2} & \end{vmatrix}$

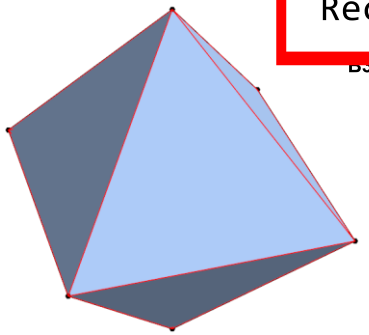
Group SO(7)

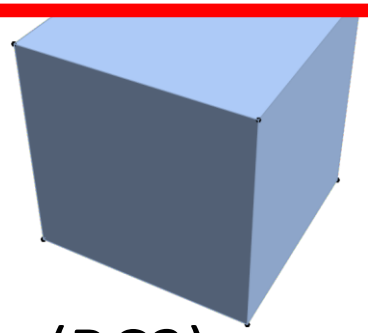
er $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 5 \\ 2 & 5 & 1 \end{pmatrix}$

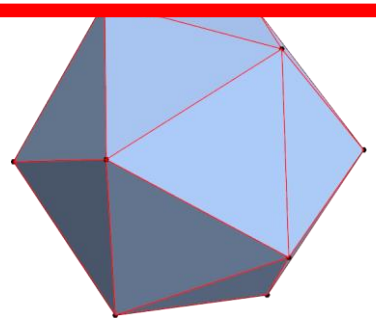
$8=2^{n=3}$ permutations of filled (aka. ringed) nodes which represent Weyl orbits that generate sets of vertices, edges, faces, and cells. The names are generated based on permutations of:

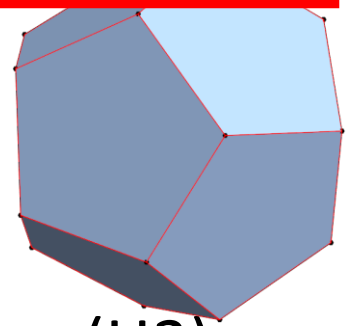
Bi	Tri	Quadri	Quinti	Hexa	Hepta
Rectified	Truncated	Cantellated	Runcinated	Stericated	Pentellated











Tetrahedron (A3)
Self Dual

Octahedron (BC3)
Cube (BC3 BiRectified)

Icosahedron (H3)
Dodecahedron (H3 BiRectified)

3D cells & polyhedra (A3, B3, H3)

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Group SU(4)

$\frac{e_1}{\sqrt{2}} + \frac{e_2}{\sqrt{2}}$ $\frac{e_3}{\sqrt{2}} - \frac{e_2}{\sqrt{2}}$ $\frac{e_2}{\sqrt{2}} - \frac{e_1}{\sqrt{2}}$

A3 Parent

Dimension=15 Rank=3 DetCM=4 # of Positive Roots=6 Coxeter #=4

CartanMatrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

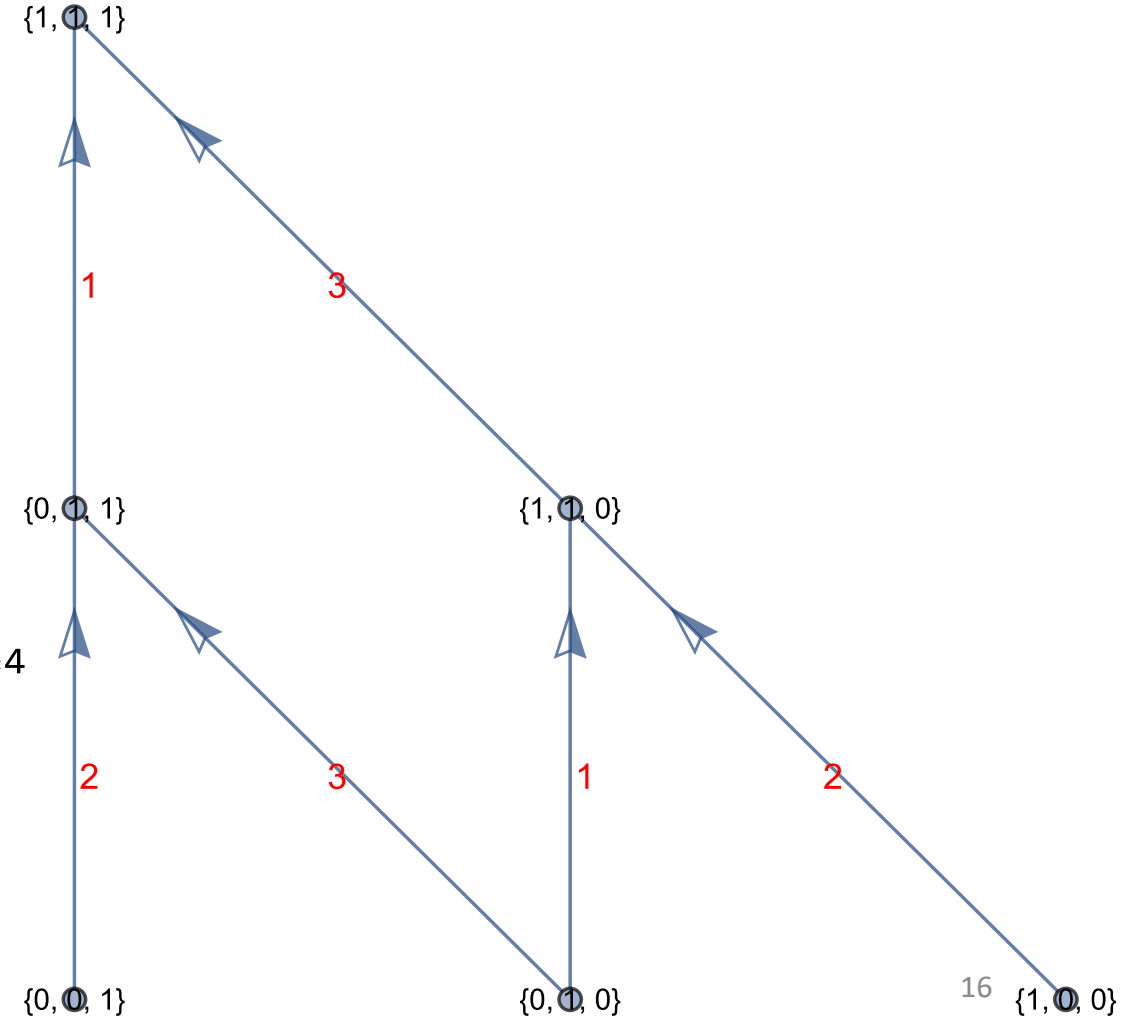
Root # $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$

Weights $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

Positive Root Vectors $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

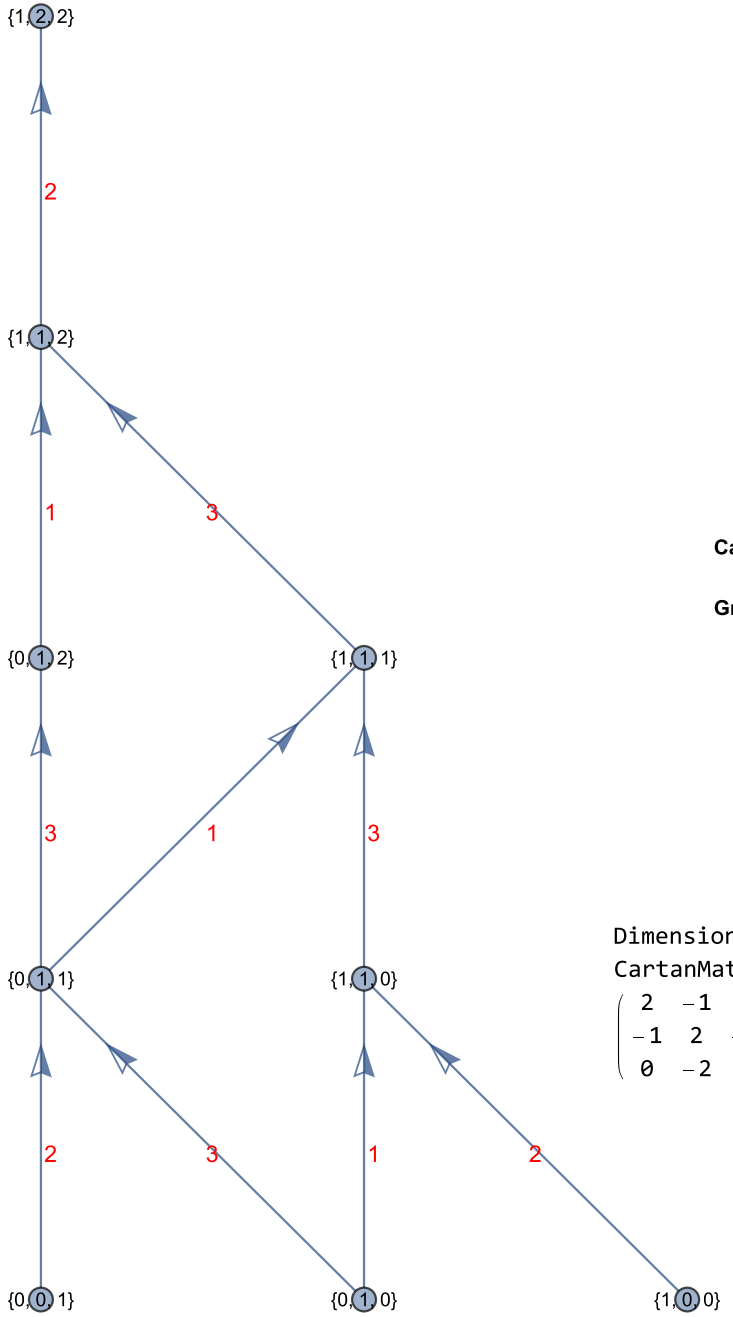
Heights $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}$

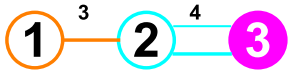
Group=SU(3)
Hasse Visualization



3D cells & polyhedra (A3, B3, H3)

Group=⊃ O(3)
Hasse Visualization



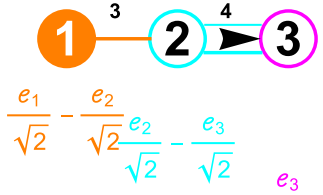


BC3 BiRectified

BC3 Parent

Cartan $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} -1 & 2 & -\sqrt{2} \\ 0 & -\sqrt{2} & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Group SO(7)



B3 Parent

Dimension=21 Rank=3 DetCM=2 # of Positive Roots=9 Coxeter #=6

CartanMatrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$

Root #	Weights	Positive Root Vectors	Heights
1	2 −1 0	1 0 0	1
2	−1 2 −2	0 1 0	1
3	0 −1 2	0 0 1	1
4	1 1 −2	1 1 0	2
5	−1 1 0	0 1 1	2
6	1 0 0	1 1 1	3
7	−1 0 2	0 1 2	3
8	1 −1 2	1 1 2	4
9	0 1 0	1 2 2	5

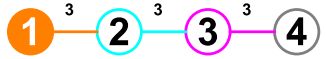
4D cells & polychora (A4, C4, D4, F4, H4)

Cartan $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}$

Group Sp(8)

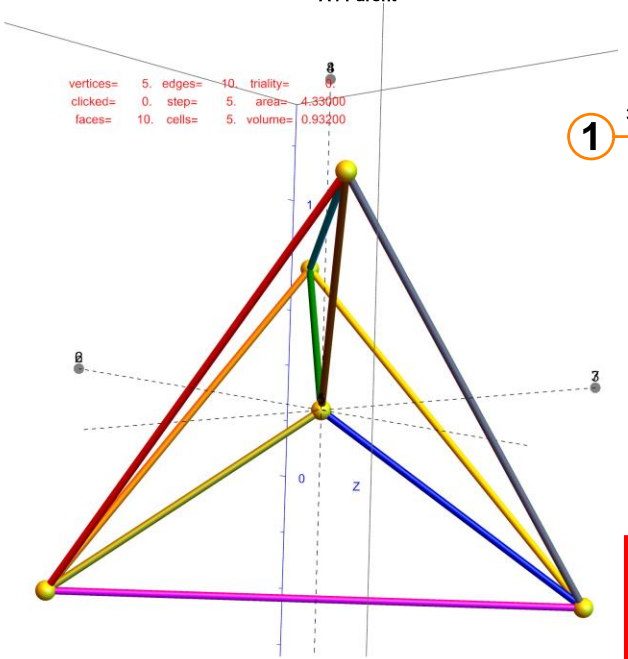
Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

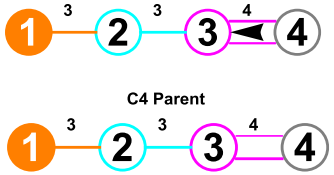
Group SU(5)



$-1 \quad \frac{e_1}{2} + \frac{e_2}{2} + \frac{e_3}{2} + \frac{1}{2} \quad -e_1 \quad \frac{e_2}{2\varphi} - \frac{\varphi e_3}{2} + \frac{e_1}{2}$

A4 Parent

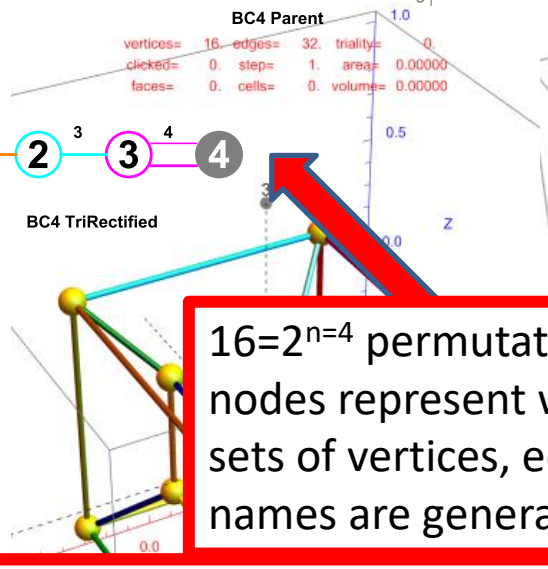


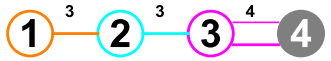


C4 Parent

$-\frac{1}{2\varphi^2} + \frac{e_2}{2\varphi} + \frac{e_3}{2} - \frac{e_2}{\varphi} - \frac{e_3}{2\varphi^2} + \frac{e_2}{2\varphi} + \frac{e_1}{2} \quad -e_1$

BC4 Parent

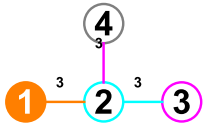




BC4 TriRectified

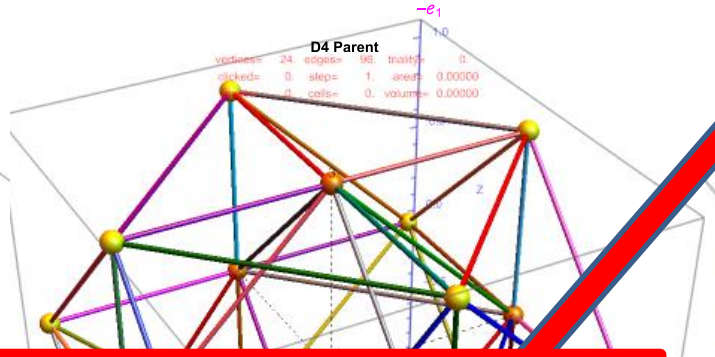
Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 3 \\ 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

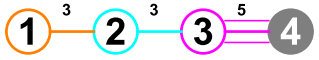
Group SO(8)



$e_2 - e_3 \quad 1 - e_1 \quad -e_2 - e_3 \quad -e_1$

D4 Parent

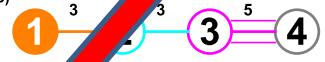




H4 TriRectified

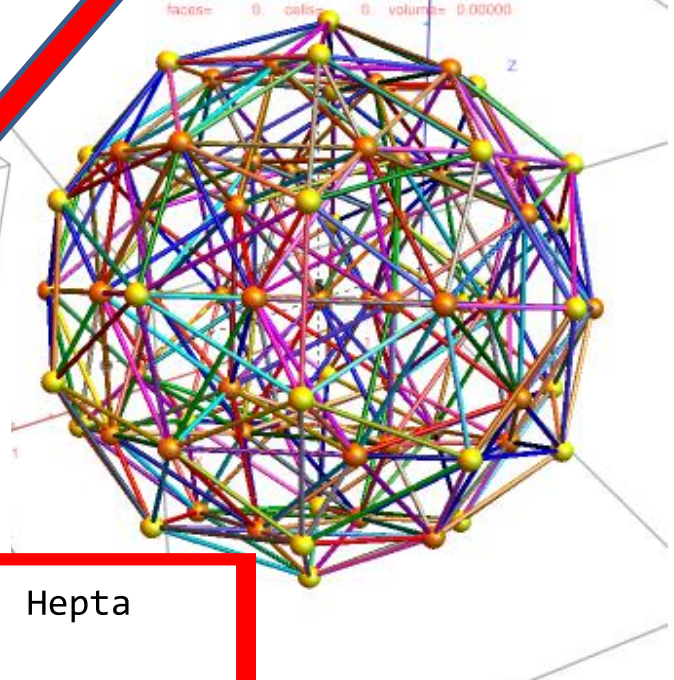
Cartan $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -\varphi \\ 0 & 0 & -\varphi & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 5 \\ 2 & 2 & 5 & 1 \end{pmatrix}$

Group ⊃ SL(2,5)



$-\frac{1}{2\varphi^2} + \frac{e_2}{2} - \frac{e_2}{\varphi} - \frac{e_3}{2\varphi^2} + \frac{e_2}{2\varphi} + \frac{e_1}{2} \quad -e_1$

H4 Parent



16=$2^{n=4}$ permutations of filled (aka. ringed) nodes represent which Weyl orbits that generate sets of vertices, edges, faces, and cells. The names are generated based on permutations of:

Bi	Tri	Quadri	Quinti	Hexa	Hepta
Rectified	Truncated	Cantellated	Runcinated	Stericated	Pentellated

5-Cell (Self Dual)

8-Cell (BC4 TriRectified aka. Tesseract)

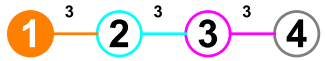
-Cell (H4)

120-Cell (H4 TriRectified)

4D cells & polychora (A4, C4, D4, F4, H4)

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Group SU(5)



$-1 \quad \frac{e_1}{2} + \frac{e_2}{2} + \frac{e_3}{2} + \frac{1}{2} \quad -e_1 \quad \frac{e_2}{2\varphi} - \frac{\varphi e_3}{2} + \frac{e_1}{2}$

A4 Parent

Dimension=24 Rank=4 DetCM=5 # of Positive Roots=10 Coxeter #=5

CartanMatrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Root #

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$$

Weights

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

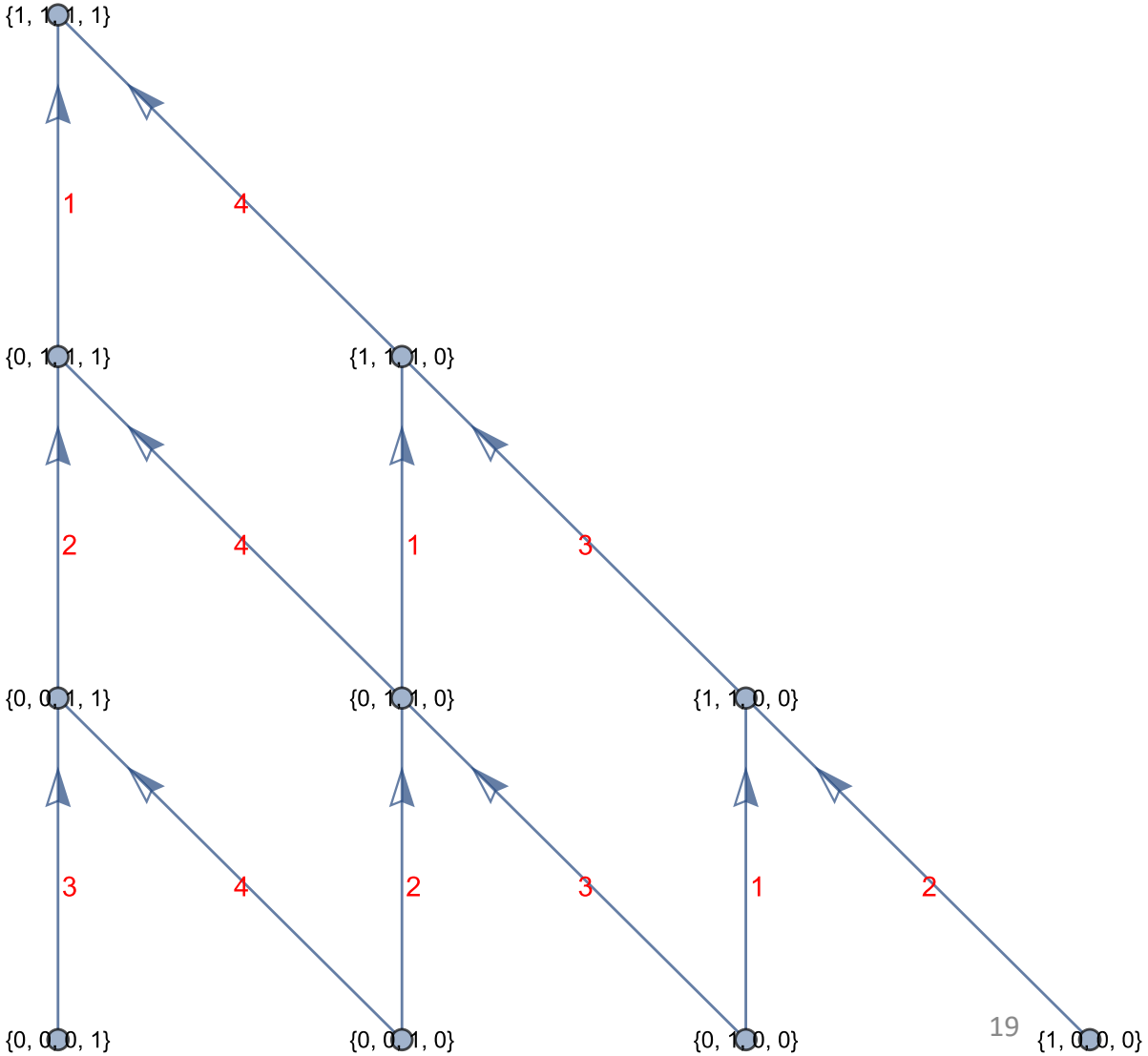
Positive Root Vectors

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Heights

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \end{pmatrix}$$

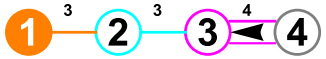
Group=SO(5)
Hasse Visualization



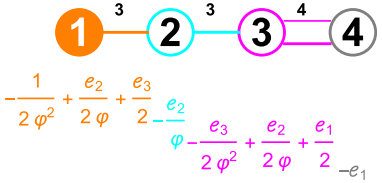
4D cells & polychora (A4, C4, D4, F4, H4)

Cartan $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\sqrt{2} \\ 0 & 0 & -\sqrt{2} & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}$

Group Sp(8)



C4 Parent



BC4 Parent

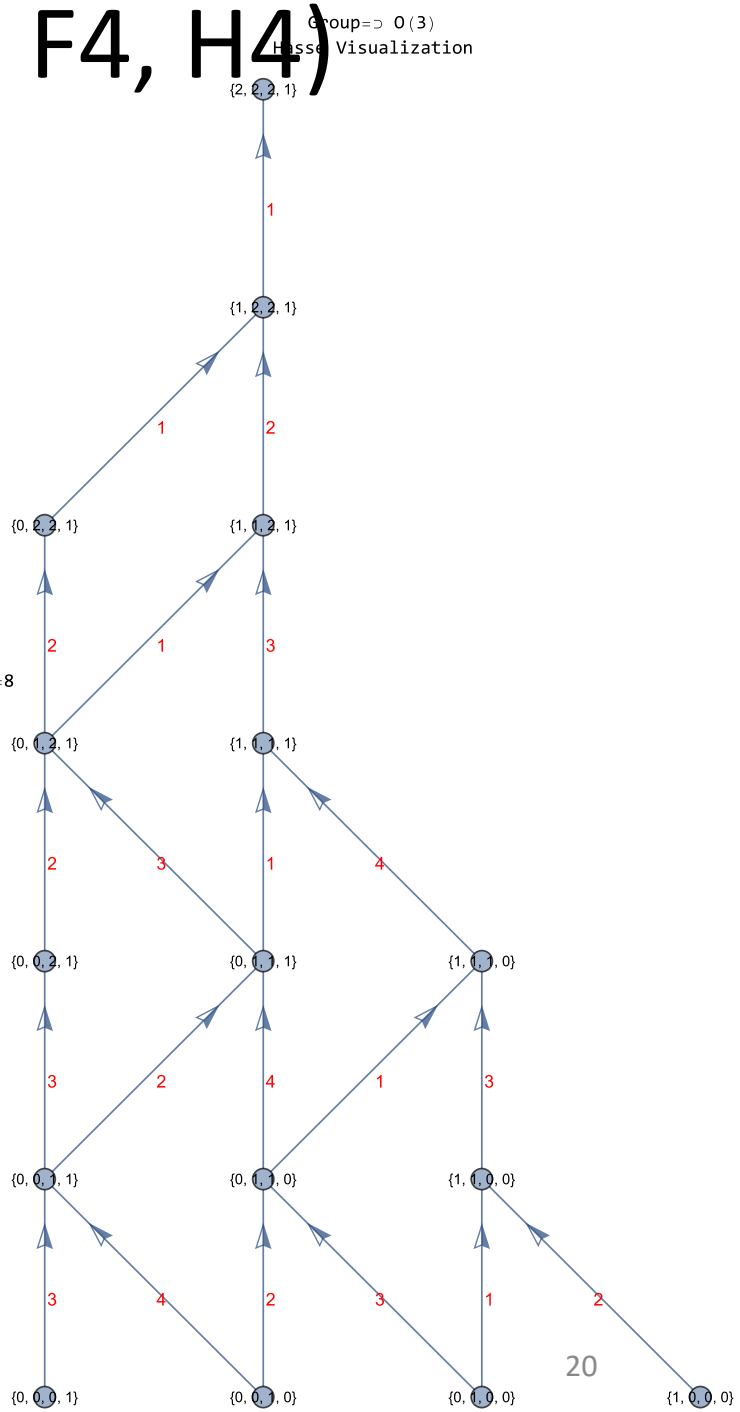
[redacted]

[redacted]

Dimension=36 Rank=4 DetCM=2 # of Positive Roots=16 Coxeter #=8

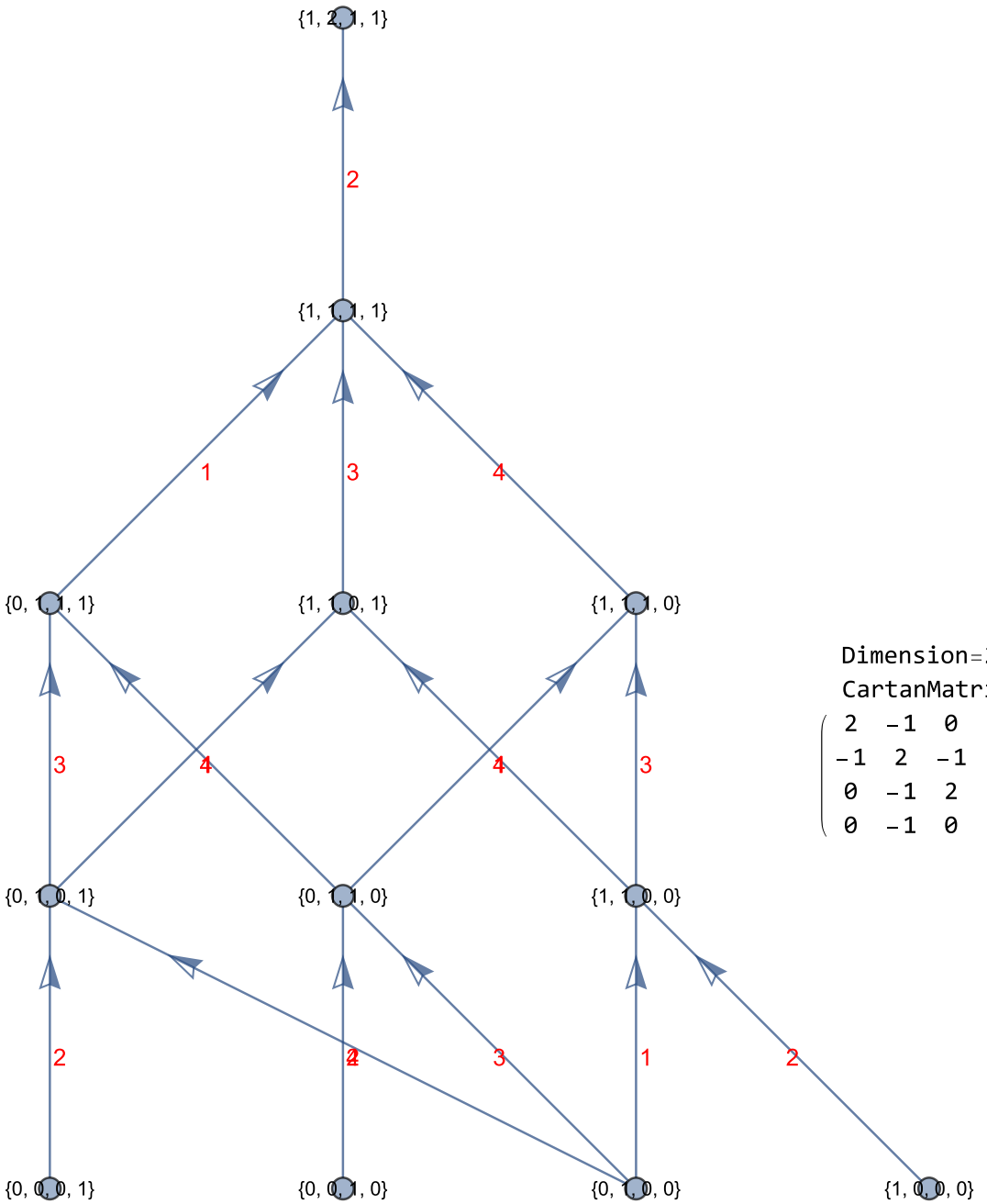
CartanMatrix $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

Root #	Weights	Positive Root Vectors	Heights
1	2 -1 0 0	1 0 0 0	1
2	-1 2 -1 0	0 1 0 0	1
3	0 -1 2 -1	0 0 1 0	1
4	0 0 -2 2	0 0 0 1	1
5	1 1 -1 0	1 1 0 0	2
6	-1 1 1 -1	0 1 1 0	2
7	0 -1 0 1	0 0 1 1	2
8	1 0 1 -1	1 1 1 0	3
9	0 -2 2 0	0 0 2 1	3
10	-1 1 -1 1	0 1 1 1	3
11	1 0 -1 1	1 1 1 1	4
12	-1 0 1 0	0 1 2 1	4
13	-2 2 0 0	0 2 2 1	5
14	1 -1 1 0	1 1 2 1	5
15	0 1 0 0	1 2 2 1	6
16	2 0 0 0	2 2 2 1	7



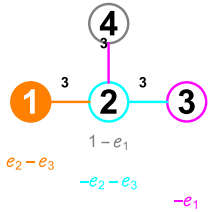
Group=SO(8)
Hasse Visualization

4D cells & polychora (A4, C4, D4, F4, H4)



Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 3 \\ 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Group SO(8)



D4 Parent

Dimension=28 Rank=4 DetCM=4 # of Positive Roots=12 Coxeter #=6

CartanMatrix $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$

Root #	Weights	Positive Root Vectors	Heights
1	2 -1 0 0	1 0 0 0	1
2	-1 2 -1 -1	0 1 0 0	1
3	0 -1 2 0	0 0 1 0	1
4	0 -1 0 2	0 0 0 1	1
5	1 1 -1 -1	1 1 0 0	2
6	-1 1 1 -1	0 1 1 0	2
7	-1 1 -1 1	0 1 0 1	2
8	1 0 1 -1	1 1 1 0	3
9	1 0 -1 1	1 1 0 1	3
10	-1 0 1 1	0 1 1 1	3
11	1 -1 1 1	1 1 1 1	4
12	0 1 0 0	1 2 1 1	5

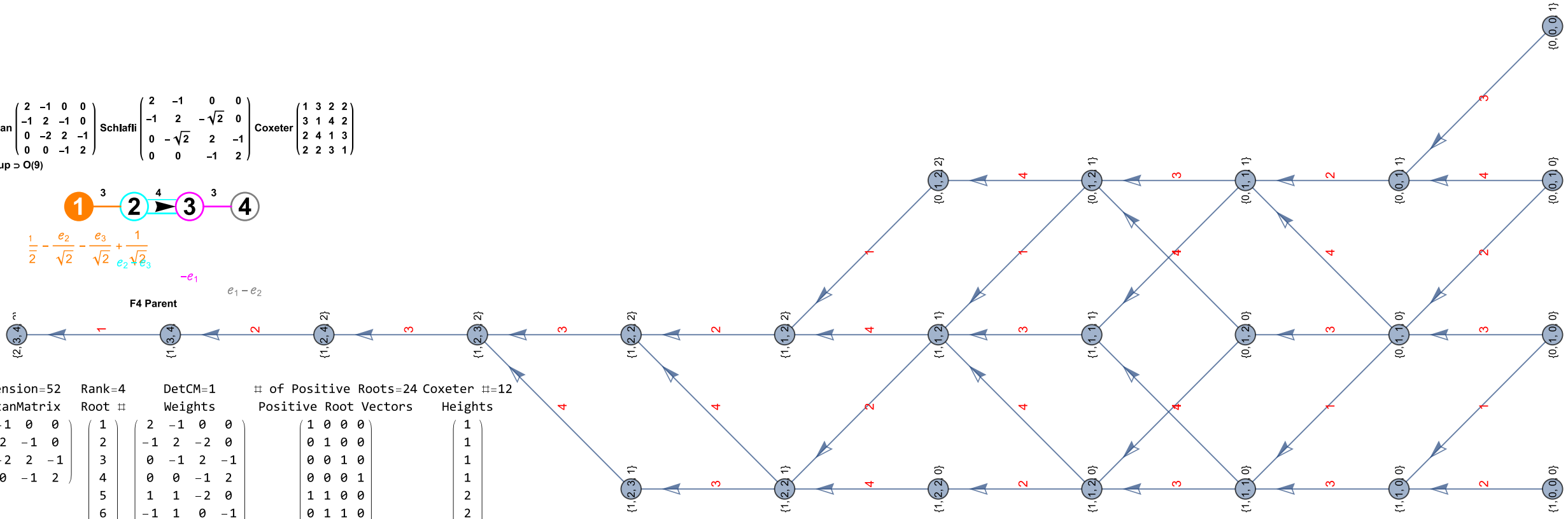
4D cells & polychora (A4, C4, D4, F4, H4)

Cartan $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}$

Group ⊃ O(9)

Group⊃ O(9)

Hasse Visualization



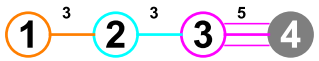
Dimension=52 Rank=4 DetCM=1 # of Positive Roots=24 Coxeter #=12

CartanMatrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Root #	Weights	Positive Root Vectors	Heights
1	2 −1 0 0	1 0 0 0	1
2	−1 2 −2 0	0 1 0 0	1
3	0 −1 2 −1	0 0 1 0	1
4	0 0 −1 2	0 0 0 1	1
5	1 1 −2 0	1 1 0 0	2
6	−1 1 0 −1	0 1 1 0	2
7	0 −1 1 1	0 0 1 1	2
8	1 0 0 −1	1 1 1 0	3
9	−1 0 2 −2	0 1 2 0	3
10	−1 1 −1 1	0 1 1 1	3
11	1 −1 2 −2	1 1 2 0	4
12	1 0 −1 1	1 1 1 1	4
13	−1 0 1 0	0 1 2 1	4
14	0 1 0 −2	1 2 2 0	5
15	1 −1 1 0	1 1 2 1	5
16	−1 0 0 2	0 1 2 2	5
17	1 −1 0 2	1 1 2 2	6
18	0 1 −1 0	1 2 2 1	6
19	0 0 1 −1	1 2 3 1	7
20	0 1 −2 2	1 2 2 2	7
21	0 0 0 1	1 2 3 2	8
22	0 −1 2 0	1 2 4 2	9
23	−1 1 0 0	1 3 4 2	10
24	1 0 0 0	2 3 4 2	11

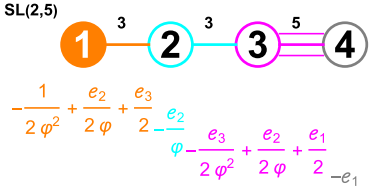
4D cells & polychora (A4, C4, D4, F4, H4)



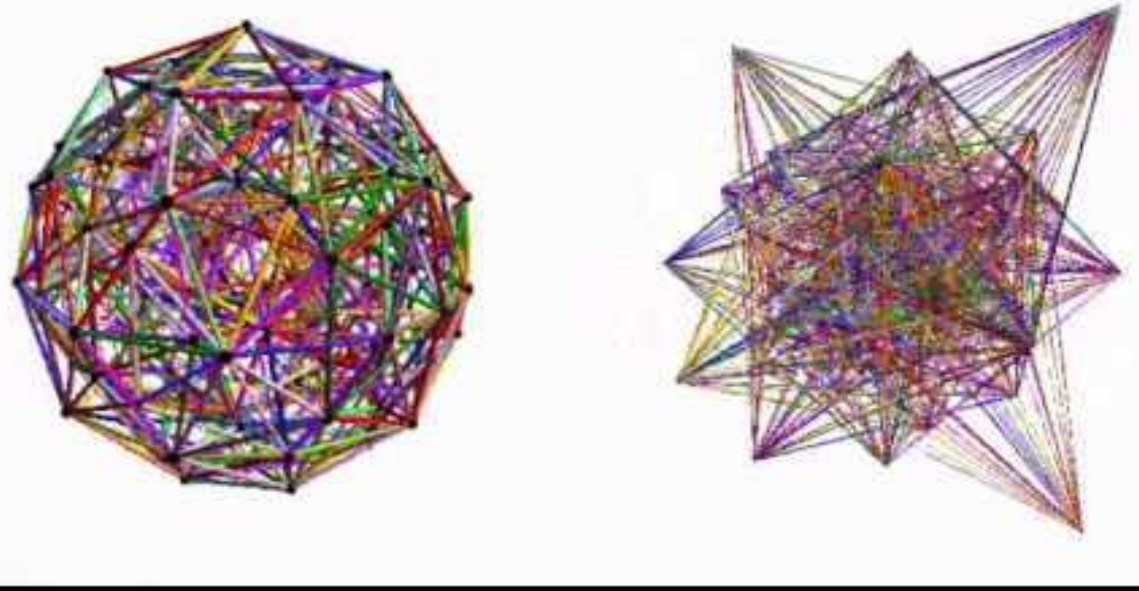
H4 TriRectified

Cartan $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\varphi \\ 0 & 0 & -\varphi & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 5 \\ 2 & 2 & 5 & 1 \end{pmatrix}$

Group ⊃ SL(2,5)



H4 Parent



5D-8D cells & polytopes
(A5, C6, D6, E6, E7, E8=BC8+D8)

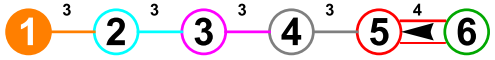
Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 3 & 1 & 3 \\ 2 & 2 & 2 & 3 & 1 \end{pmatrix}$

Group SU(6)

A5 Parent

Cartan $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$ Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 & 1 & 4 \\ 2 & 2 & 2 & 2 & 4 & 1 \end{pmatrix}$

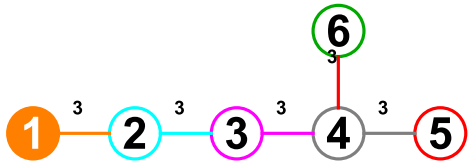
Group Sp(12)



C6 Parent

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 3 \\ 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 2 & 3 & 2 & 1 \end{pmatrix}$

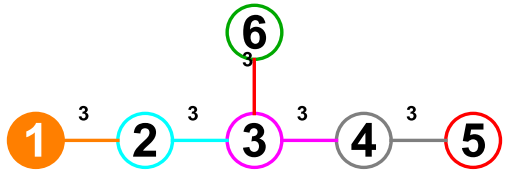
Group SO(12)



D6 Parent

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 3 \\ 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 3 & 2 & 2 & 1 \end{pmatrix}$

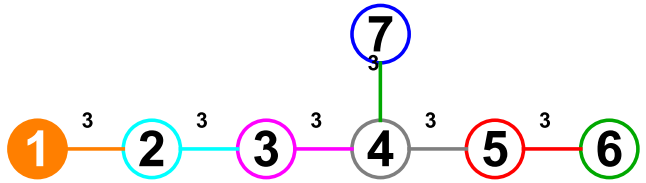
Group ⊃ O(12)



E6 Parent

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 3 & 2 & 2 & 2 & 1 \end{pmatrix}$

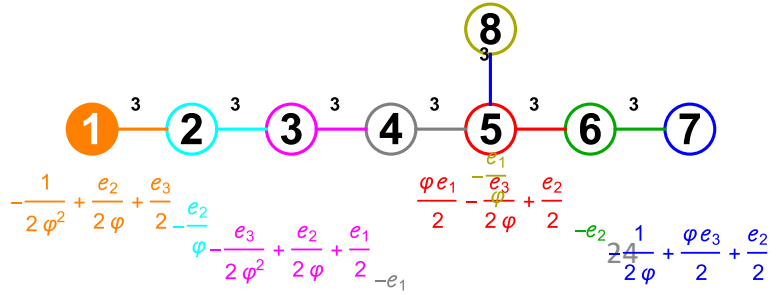
Group ⊃ O(14)



E7 Parent

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 2 & 2 \\ 2 & 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & 3 & 1 \end{pmatrix}$ Simple Roots $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$

Group ⊃ O(16)



E8 Parent

5D-8D cells & polytopes
(A5, C6, D6, E6, E7, E8=BC8+D8)

Cartan=Schlafli $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$ Coxeter $\begin{pmatrix} 1 & 3 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 3 & 1 & 3 \\ 2 & 2 & 2 & 3 & 1 \end{pmatrix}$

Group SU(6)

A5 Parent

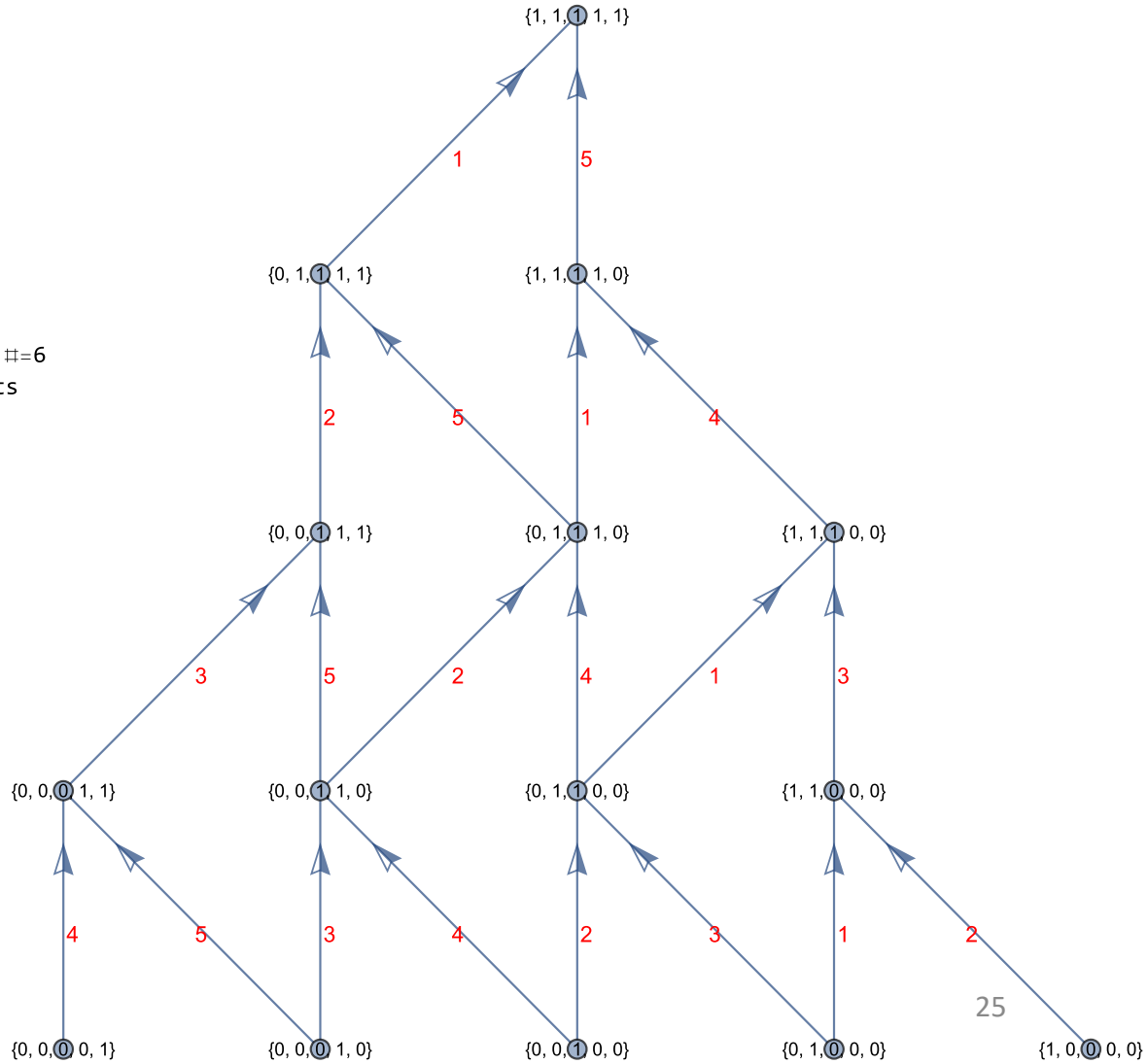
Dimension=35 Rank=5 DetCM=6 # of Positive Roots=15 Coxeter #=6

CartanMatrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

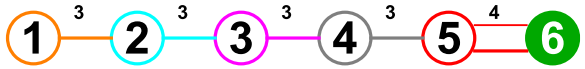
Root #	Weights					Positive Root Vectors					Heights
1	2	-1	0	0	0	1	0	0	0	0	1
2	-1	2	-1	0	0	0	1	0	0	0	1
3	0	-1	2	-1	0	0	0	1	0	0	1
4	0	0	-1	2	-1	0	0	0	1	0	1
5	0	0	0	-1	2	0	0	0	0	1	1
6	1	1	-1	0	0	1	1	0	0	0	2
7	-1	1	1	-1	0	0	1	1	0	0	2
8	0	-1	1	1	-1	0	0	1	1	0	2
9	0	0	-1	1	1	0	0	0	1	1	2
10	1	0	1	-1	0	1	1	1	0	0	3
11	-1	1	0	1	-1	0	1	1	1	0	3
12	0	-1	1	0	1	0	0	1	1	1	3
13	1	0	0	1	-1	1	1	1	1	0	4
14	-1	1	0	0	1	0	1	1	1	1	4
15	1	0	0	0	1	1	1	1	1	1	5

Group=SO(5)
Hasse Visualization

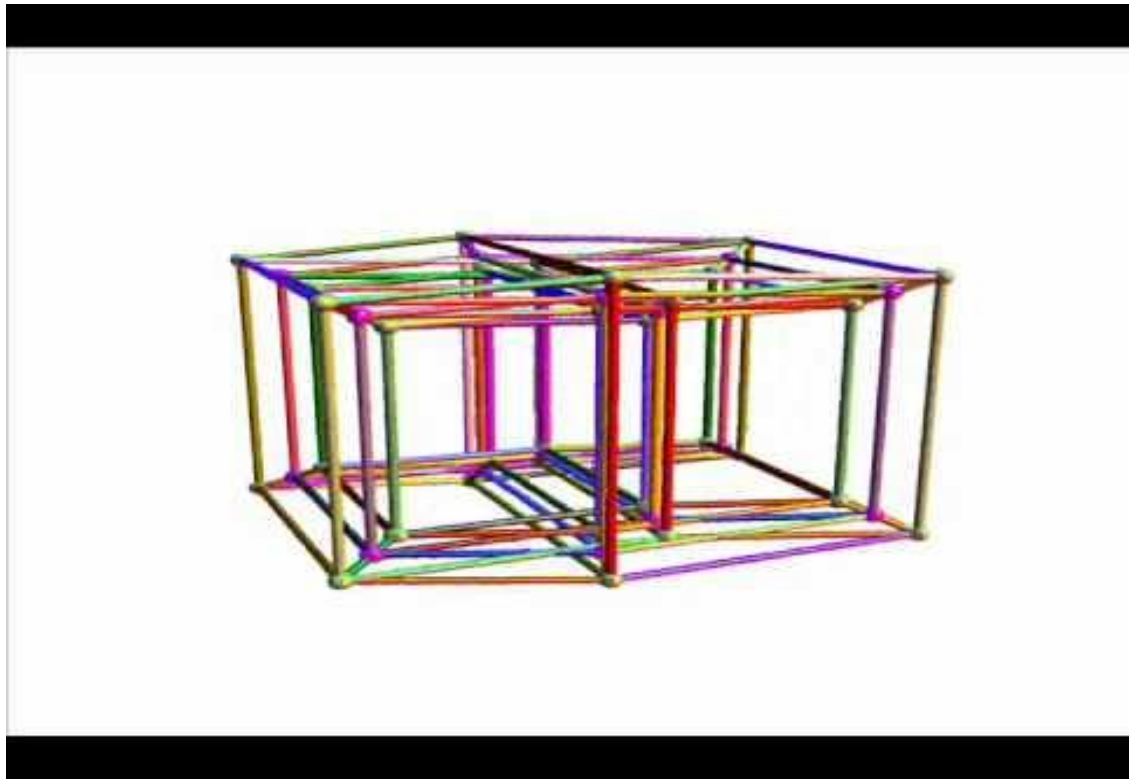


5D-8D cells & polytopes
(A5, C6, D6, E6, E7, E8=BC8+D8)

BC6 Hexeract



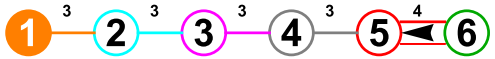
BC6 QuintiRectified



Cartan
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$
Group Sp(12)

Schlafli
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 2 \end{pmatrix}$$

Coxeter
$$\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 & 1 & 4 \\ 2 & 2 & 2 & 2 & 4 & 1 \end{pmatrix}$$

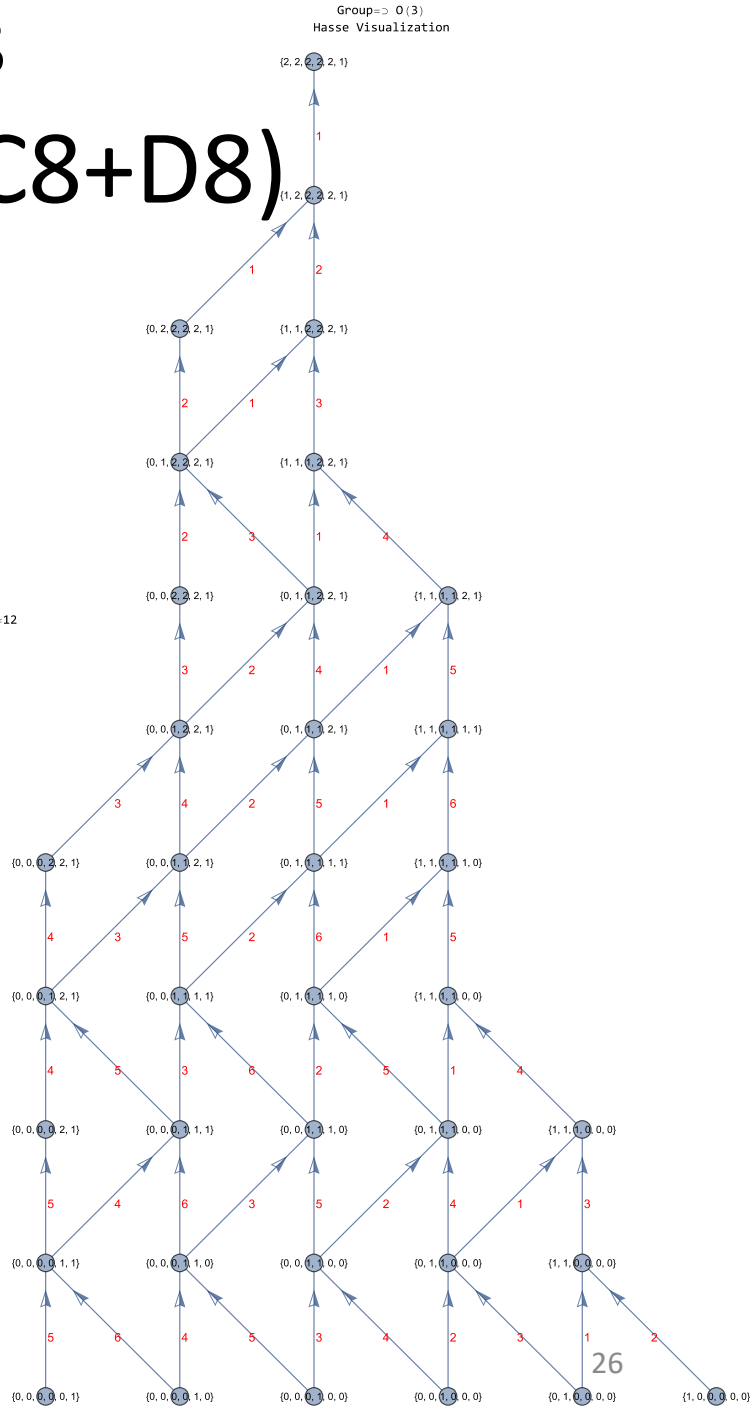


C6 Parent

Dimension=78 Rank=6 DetCM=2 # of Positive Roots=36 Coxeter #=12

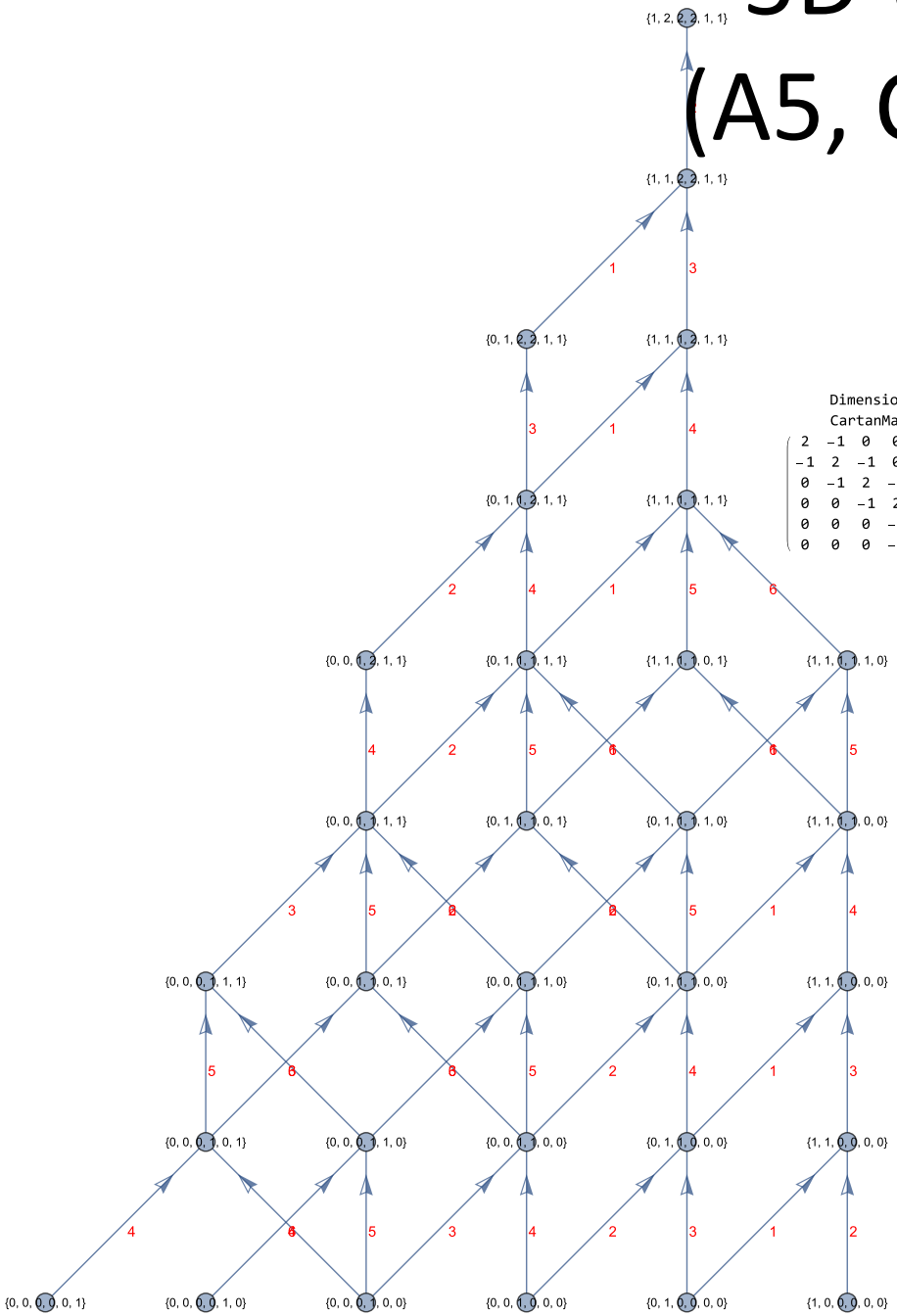
CartanMatrix
$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Root #	Weights	Positive Root Vectors	Heights
1	2 -1 0 0 0 0	1 0 0 0 0 0	1
2	-1 2 -1 0 0 0	0 1 0 0 0 0	1
3	0 -1 2 -1 0 0	0 0 1 0 0 0	1
4	0 0 -1 2 -1 0	0 0 0 1 0 0	1
5	0 0 0 -1 2 -1	0 0 0 0 1 0	1
6	0 0 0 0 -2 2	0 0 0 0 0 1	1
7	1 1 -1 0 0 0	1 1 0 0 0 0	2
8	-1 1 1 -1 0 0	0 1 1 0 0 0	2
9	0 -1 1 1 -1 0	0 0 1 1 0 0	2
10	0 0 -1 1 1 -1	0 0 0 1 1 0	2
11	0 0 0 -1 0 1	0 0 0 0 1 1	2
12	1 0 1 -1 0 0	1 1 1 0 0 0	3
13	-1 1 0 1 -1 0	0 1 1 1 0 0	3
14	0 -1 1 0 1 -1	0 0 1 1 1 0	3
15	0 0 0 -2 2 0	0 0 0 0 2 1	3
16	0 0 -1 1 -1 1	0 0 0 1 1 1	3
17	1 0 0 1 -1 0	1 1 1 1 0 0	4
18	-1 1 0 0 1 -1	0 1 1 1 1 0	4
19	0 -1 1 0 -1 1	0 0 1 1 1 1	4
20	0 0 -1 0 1 0	0 0 0 1 2 1	4
21	1 0 0 0 1 -1	1 1 1 1 1 0	5
22	0 0 -2 2 0 0	0 0 0 2 2 1	5
23	-1 1 0 0 -1 1	0 1 1 1 1 1	5
24	0 -1 1 -1 1 0	0 0 1 1 2 1	5
25	1 0 0 0 -1 1	1 1 1 1 1 1	6
26	-1 1 0 -1 1 0	0 1 1 1 2 1	6
27	0 -1 0 1 0 0	0 0 1 2 2 1	6
28	0 -2 2 0 0 0	0 0 2 2 2 1	7
29	1 0 0 -1 1 0	1 1 1 1 2 1	7
30	-1 1 -1 1 0 0	0 1 1 2 2 1	7
31	1 0 -1 1 0 0	1 1 1 2 2 1	8
32	-1 0 1 0 0 0	0 1 2 2 2 1	8
33	-2 2 0 0 0 0	0 2 2 2 2 1	9
34	1 -1 1 0 0 0	1 1 2 2 2 1	9
35	0 1 0 0 0 0	1 2 2 2 2 1	10
36	2 0 0 0 0 0	2 2 2 2 2 1	11



5D-8D cells & polytopes (A5, C6, D6, E6, E7, E8=BC8+D8)

Group=Sp(12)
Hasse Visualization



Dimension=66 Rank=6 DetCM=4 # of Positive Roots=30 Coxeter #=10

CartanMatrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Root #	Weights	Positive Root Vectors	Heights
1	2 -1 0 0 0 0	1 0 0 0 0 0	1
2	-1 2 -1 0 0 0	0 1 0 0 0 0	1
3	0 -1 2 -1 0 0	0 0 1 0 0 0	1
4	0 0 -1 2 -1 -1	0 0 0 1 0 0	1
5	0 0 0 -1 2 0	0 0 0 0 1 0	1
6	0 0 0 -1 0 2	0 0 0 0 0 1	1
7	1 1 -1 0 0 0	1 1 0 0 0 0	2
8	-1 1 1 -1 0 0	0 1 1 0 0 0	2
9	0 -1 1 1 -1 -1	0 0 1 1 0 0	2
10	0 0 -1 1 1 -1	0 0 0 1 1 0	2
11	0 0 -1 1 -1 1	0 0 0 1 0 1	2
12	1 0 1 -1 0 0	1 1 1 0 0 0	3
13	-1 1 0 1 -1 -1	0 1 1 1 0 0	3
14	0 -1 1 0 1 -1	0 0 1 1 1 0	3
15	0 -1 1 0 -1 1	0 0 1 1 0 1	3
16	0 0 -1 0 1 1	0 0 0 1 1 1	3
17	0 -1 1 -1 1 1	0 0 1 1 1 1	4
18	1 0 0 1 -1 -1	1 1 1 1 0 0	4
19	-1 1 0 0 1 -1	0 1 1 1 1 0	4
20	-1 1 0 0 -1 1	0 1 1 1 0 1	4
21	-1 1 0 -1 1 1	0 1 1 1 1 1	5
22	1 0 0 0 1 -1	1 1 1 1 1 0	5
23	1 0 0 0 -1 1	1 1 1 1 0 1	5
24	0 -1 0 1 0 0	0 0 1 2 1 1	5
25	1 0 0 -1 1 1	1 1 1 1 1 1	6
26	-1 1 -1 1 0 0	0 1 1 2 1 1	6
27	-1 0 1 0 0 0	0 1 2 2 1 1	7
28	1 0 -1 1 0 0	1 1 1 2 1 1	7
29	1 -1 1 0 0 0	1 1 2 2 1 1	8
30	0 1 0 0 0 0	1 2 2 2 1 1	9

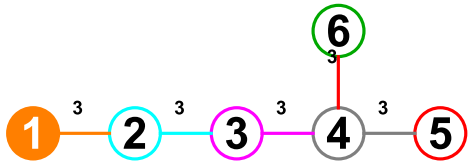
Cartan=Schlafli

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Coxeter

$$\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 3 \\ 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 2 & 3 & 2 & 1 \end{pmatrix}$$

Group SO(12)



D6 Parent

5D-8D cells & polytopes
(A5, C6, D6, E6, E7, E8=BC8+D8)

Dimension=78 Rank=6 DetCM=3 # of Positive Roots=36 Coxeter #=12

CartanMatrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

Root #	Weights	Positive Root Vectors	Heights
1	2 −1 0 0 0 0	1 0 0 0 0 0	1
2	−1 2 −1 0 0 0	0 1 0 0 0 0	1
3	0 −1 2 −1 0 −1	0 0 1 0 0 0	1
4	0 0 −1 2 −1 0	0 0 0 1 0 0	1
5	0 0 0 −1 2 0	0 0 0 0 1 0	1
6	0 0 −1 0 0 2	0 0 0 0 0 1	1
7	1 1 −1 0 0 0	1 1 0 0 0 0	2
8	−1 1 1 −1 0 −1	0 1 1 0 0 0	2
9	0 −1 1 1 −1 −1	0 0 1 1 0 0	2
10	0 −1 1 −1 0 1	0 0 1 0 0 1	2
11	0 0 −1 1 1 0	0 0 0 1 1 0	2
12	1 0 1 −1 0 −1	1 1 1 0 0 0	3
13	−1 1 0 1 −1 −1	0 1 1 1 0 0	3
14	0 −1 1 0 1 −1	0 0 1 1 1 0	3
15	−1 1 0 −1 0 1	0 1 1 0 0 1	3
16	0 −1 0 1 −1 1	0 0 1 1 0 1	3
17	−1 1 −1 1 −1 1	0 1 1 1 0 1	4
18	1 0 0 1 −1 −1	1 1 1 1 0 0	4
19	1 0 0 −1 0 1	1 1 1 0 0 1	4
20	−1 1 0 0 1 −1	0 1 1 1 1 0	4
21	0 −1 0 0 1 1	0 0 1 1 1 1	4
22	1 0 −1 1 −1 1	1 1 1 1 0 1	5
23	−1 0 1 0 −1 0	0 1 2 1 0 1	5
24	1 0 0 0 1 −1	1 1 1 1 1 0	5
25	−1 1 −1 0 1 1	0 1 1 1 1 1	5
26	1 0 −1 0 1 1	1 1 1 1 1 1	6
27	1 −1 1 0 −1 0	1 1 2 1 0 1	6
28	−1 0 1 −1 1 0	0 1 2 1 1 1	6
29	1 −1 1 −1 1 0	1 1 2 1 1 1	7
30	0 1 0 0 −1 0	1 2 2 1 0 1	7
31	−1 0 0 1 0 0	0 1 2 2 1 1	7
32	1 −1 0 1 0 0	1 1 2 2 1 1	8
33	0 1 0 −1 1 0	1 2 2 1 1 1	8
34	0 1 −1 1 0 0	1 2 2 2 1 1	9
35	0 0 1 0 0 −1	1 2 3 2 1 1	10
36	0 0 0 0 0 1	1 2 3 2 1 2	11

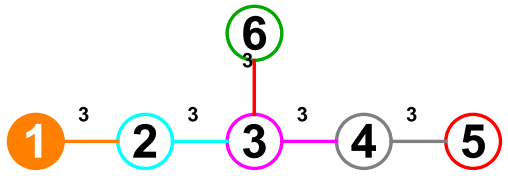
Cartan=Schlafli

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

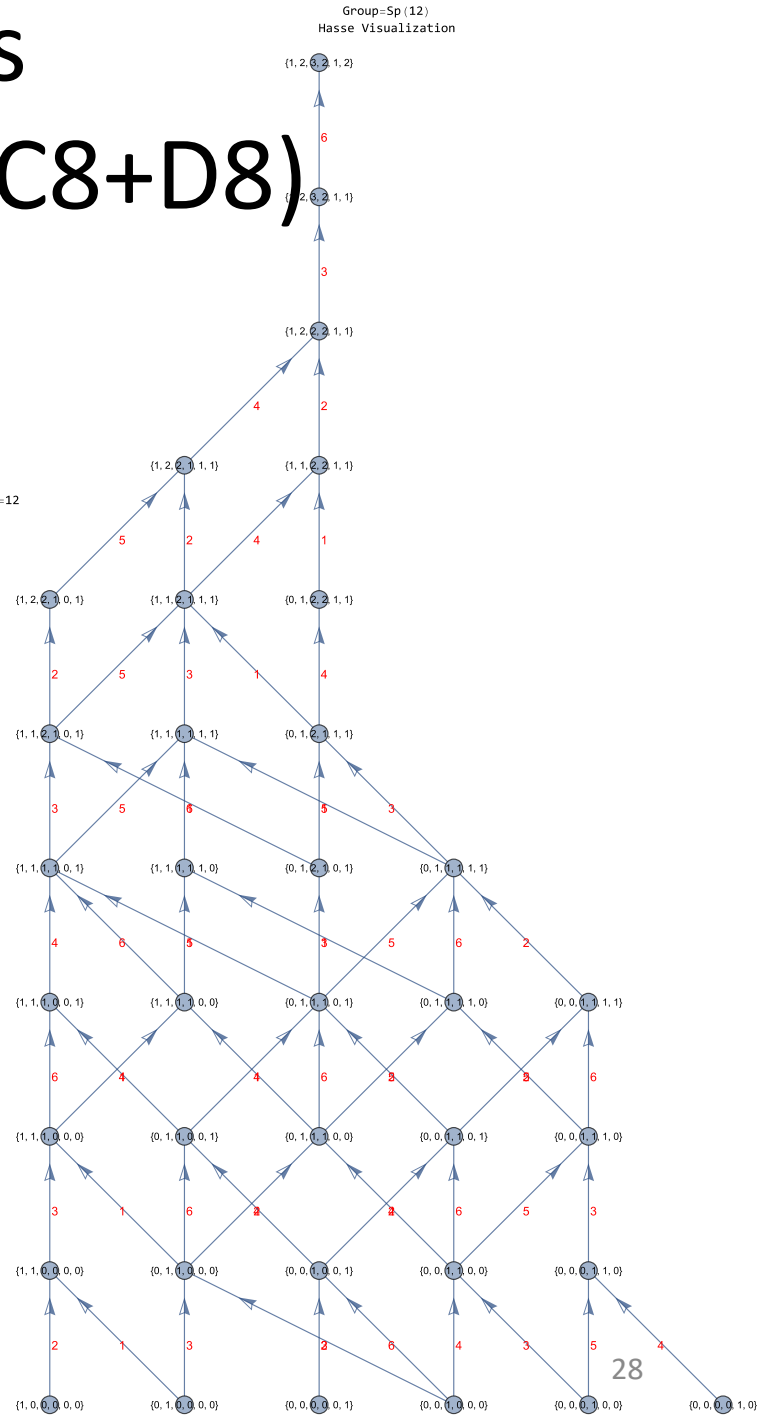
Coxeter

$$\begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 3 \\ 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 3 & 2 & 2 & 1 \end{pmatrix}$$

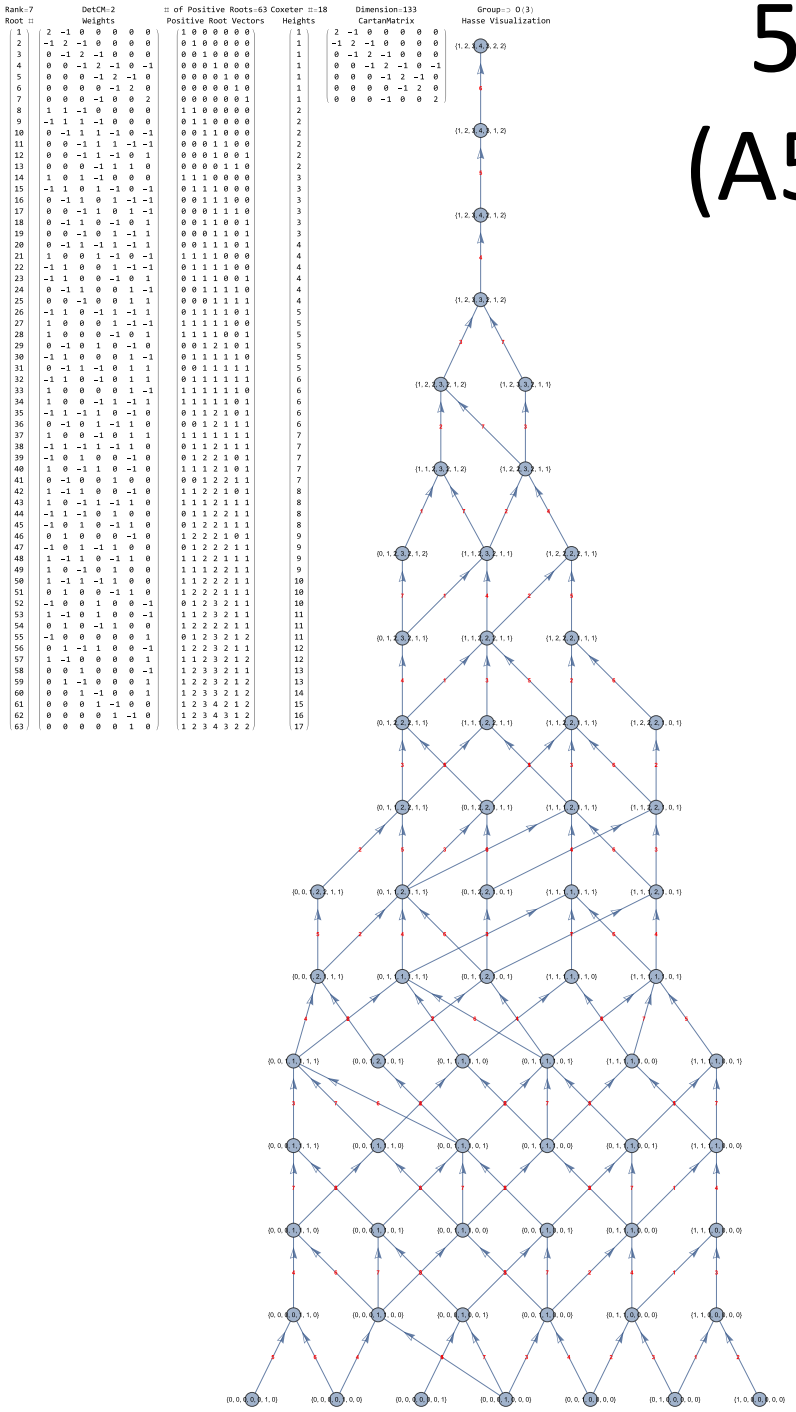
Group ⊃ O(12)



E6 Parent

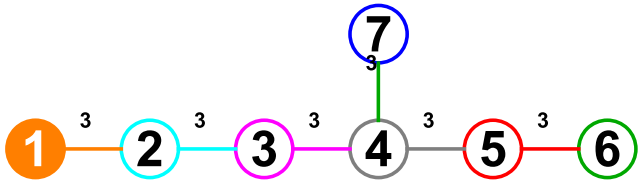


5D-8D cells & polytopes
(A5, C6, D6, E6, E7, E8=BC8+D8)



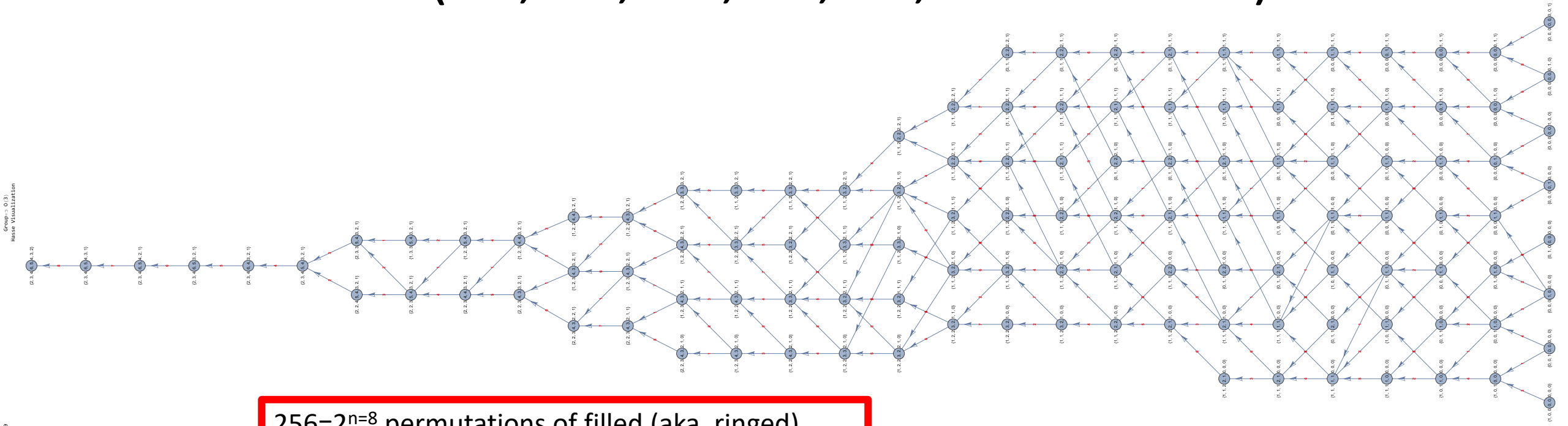
$$\text{Cartan=Schlafli}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \text{Coxeter} \begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 2 & 3 \\ 2 & 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 2 & 3 & 2 & 2 & 1 \end{pmatrix}$$

Group ⊃ O(14)



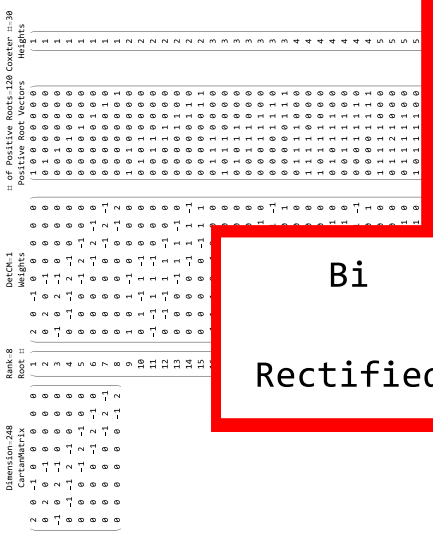
E7 Parent

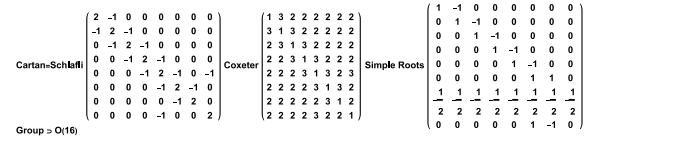
5D-8D cells & polytopes
(A5, C6, D6, E6, E7, E8=BC8+D8)

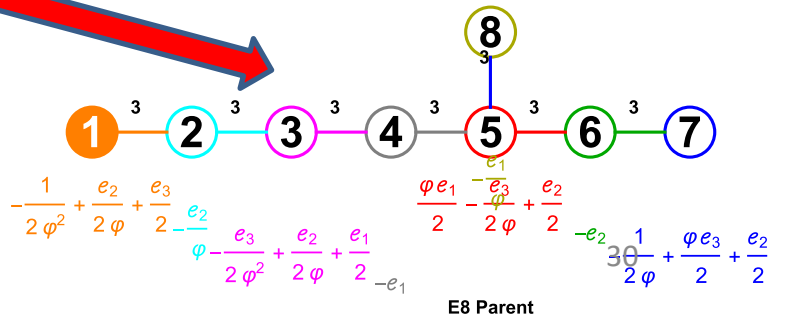


256=$2^{n=8}$ permutations of filled (aka. ringed) nodes which represent Weyl orbits that generate sets of vertices, edges, faces, and cells. The names are generated based on:

Bi	Tri	Quadri	Quinti	Hexa	Hepta
Rectified	Truncated	Cantellated	Runcinated	Stericated	Pentellated





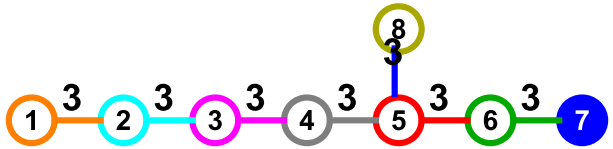
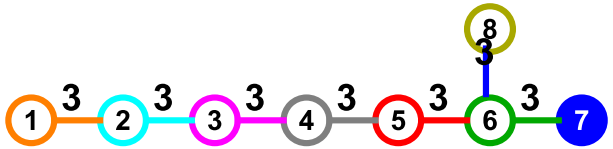


5D-8D cells & polytopes

(A5, C6, D6, E6, E7, E8=BC8+D8)

SO16 Projection Matrix

$$\begin{pmatrix} -2 & -4 & -6 & -5 & -4 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Group Coxeter–Dynkin Diagram	Coxeter–Dynkin Weyl Orbit	E8 Rep	SO16 Content	Subgroup Coxeter–Dynkin Diagrams
 E8 HexaRectified	{0, 0, 0, 0, 0, 0, 1, 0}	248	{120, 128'}	 D8 HexaRectified

E8 Maximal Embeddings

BC8

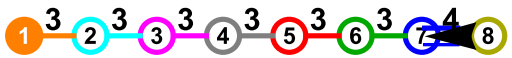
SO(16)=D8 Height 120=(112+4+4)+128’ .

5D-8D cells & polytopes

(A5, C6, D6, E6, E7, E8=BC8+D8)

SP8⊗SP8 Projection Matrix

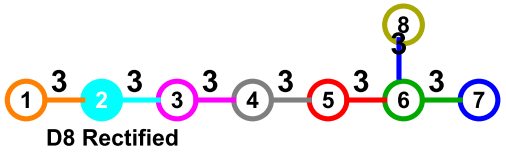
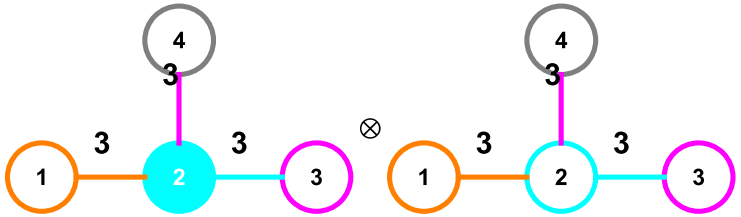
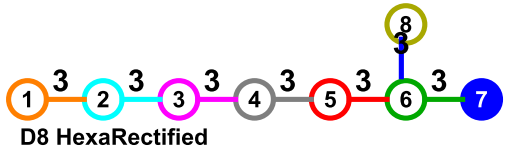
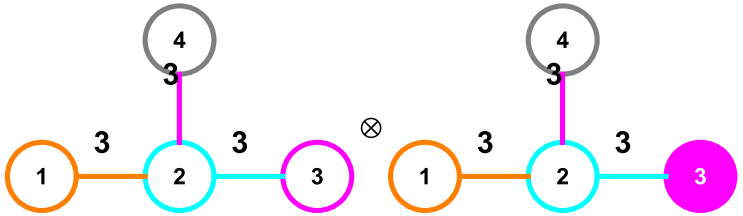
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Group Coxeter–Dynkin Diagram	Coxeter–Dynkin Weyl Orbit	SP16 Rep	SP8⊗SP8 Content	Subgroup Coxeter–Dynkin Diagrams
 C8 Parent	{2, 0, 0, 0, 0, 0, 0, 0}	136	$\{36\otimes 1,\ 1\otimes 36,\ 8^{\otimes 2}\}$ 4 + 4	 C4 Parent ⊗ C4 Snub

C8 Maximal Embeddings Height 128+4+4

SO8⊗SO8 Projection Matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -2 & -2 & -2 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

 D8 Rectified	{0, 1, 0, 0, 0, 0, 0, 0}	1	$\{28\otimes 1,\ 1\otimes 28,\ 8_v^{\otimes 2}\}$ 4 + 4	 D4 Rectified ⊗ D4 Snub
 D8 HexaRectified	{0, 0, 0, 0, 0, 0, 1, 0}	128' +4+4	$\{8_c\otimes 8_s,\ 8_s\otimes 8_c\}$	 D4 Snub ⊗ D4 BiRectified

D8 Maximal Embeddings Height 112+4+4

Agenda

- Introducing the Mathematica™ based VisibLie-E8 Tool
  - The Coxeter-Dynkin Visualizer (Groups, Weyl Orbits, Hasse Diagrams, etc.)
  - The (Bi)Quaternion/(Bi)Octonion/Sedenion Engine (Fano Plane/Cube/Tesseract, etc.)
  - The VisibLie_E8 Hyperdimensional Visualizer (2D/3D Projections, etc.)
- Introducing the Dimensions (0 to 8, ∞)
  - 0D points, 1D edges & 2D faces (A2, B2=C2, G2, H2 Groups)
  - 3D cells & polyhedra (A3, B3, H3 Groups)
  - 4D cells & polychora (A4, C4, D4, F4, H4 Groups)
  - 5D-8D cells & polytopes (A5, C6, D6, E6, E7, E8=BC8+D8 Groups)
- Generating Solid Convex Hulls from Quaternions and their Weyl Group Orbits
  - The 5 Platonic Solids, which includes duals
  - The 13 Archimedean Solids and their Catalan Duals, with some Johnson and Near-miss Johnsons
- 4D Polychora w/Duals, including the Dual Snub 24-Cell

Generating Solid Convex Hulls from Quaternions and their Weyl Group Orbits

Weyl orbits [edit]

Another construction method uses quaternions and the Icosahedral symmetry of Weyl group orbits $O(\Lambda) = W(H_4) = I$ of order 120.[18] The following describe T and T' 24-cells as quaternion orbit weights of D4 under the Weyl group W(D4):

O(0100) : T = {±1,±e1,±e2,±e3,(±1±e1±e2±e3)/2}

O(1000) : V1

O(0010) : V2

O(0001) : V3

$$T' = \sqrt{2}\ \{V1 \oplus V2 \oplus V3\} = \begin{pmatrix} \frac{-1-e1}{\sqrt{2}} & \frac{1-e1}{\sqrt{2}} & \frac{-1+e1}{\sqrt{2}} & \frac{1+e1}{\sqrt{2}} & \frac{-e2-e3}{\sqrt{2}} & \frac{e2-e3}{\sqrt{2}} & \frac{-e2+e3}{\sqrt{2}} & \frac{e2+e3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1-e2}{\sqrt{2}} & \frac{-1+e2}{\sqrt{2}} & \frac{1+e2}{\sqrt{2}} & \frac{-e1-e3}{\sqrt{2}} & \frac{e1-e3}{\sqrt{2}} & \frac{-e1+e3}{\sqrt{2}} & \frac{e1+e3}{\sqrt{2}} \\ \frac{-e1-e2}{\sqrt{2}} & \frac{e}{\sqrt{2}} & \frac{-e1+e2}{\sqrt{2}} & & & & & \end{pmatrix};$$

With quaternions (p, q) where $\bar{p}$ is the conjugate ... ar
$W(H_4) = \{[p, \bar{p}] \oplus [p, \bar{p}]^*\}$ is the symmetry group

Given $p \in T$ such that $\bar{p} = \pm p^4, \bar{p}^2 = \pm p^3, \bar{p}^3 = \pm p$

- the snub 24-cell $S = \sum_{i=1}^{4} \oplus p^i T$
- the 600-cell $I = T + S = \sum_{i=0}^{4} \oplus p^i T$
- the 120-cell $J = \sum_{i,j=0}^{4} \oplus p^i \bar{p}^{\dagger j} T'$
- the dual snub 24-cell = $T \oplus T' \oplus S'$.

This is from the Wikipedia page describing the mathematics behind the 4D quaternion polytopes generated in this presentation.

D4

T =Rectified (if node 2 is in the middle of D4 or Parent if it is on left)

V1 =Parent (if node 2 is in the middle of D4 or Rectified if it is on left)

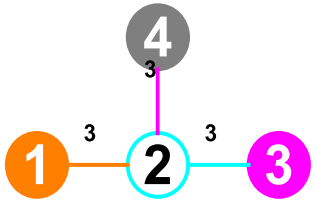
V2 =BiRectified

V3 =TriRectified

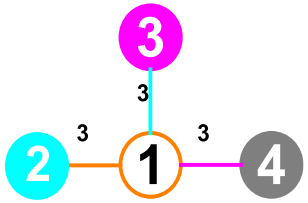
T' =CantiRuncinated (if node 2 is in the middle of D4 or BiCantiTruncated if it is on left)

T+T'=OmniTruncated

with Snub=O(0000)



D4 CantiRuncinated



D4 BiCantiTruncated

Generating Solid Convex Hulls from Quaternions and their Weyl Group Orbits

```
In[•]:= (* Arbitrary Integers a1,a2,a3 that get assigned from Weyl orbital weights *)
α = (a1 - a3) / 2;
β = (a1 + a3) / 2;
γ = (a1 + 2 a2 + a3) / 2;

a := {a1, a2, a3};
```

These are snippets of my Mathematica implementation which supports the generation of the output shown.

```
In[•]:= (* Resolving the {a1,a2,a3} using Weyl orbit weights *)
aRule@select_List := Inner[Rule, a, select, List];
aRule@{0, 0, 1}

Out[•]= {a1 → 0, a2 → 0, a3 → 1}

(* Quaternion prq function *)
r1@q_ := prq[α1, q*, -α1];
r2@q_ := prq[α2, q*, -α2];
r3@q_ := prq[α3, q*, -α3];

In[•]:= Λ[f_, in_List, select_List : {0, 0, 0}, permType_ : True] := f[in /. aRule@select, permType];

(* The Tribonacci Constant - Using 3 consecutive numbers ({0,0,1} vs. 2 consecutive {0,1}*)
xRep = ToRadicals@Solve[x^3 - x^2 - x - 1 == 0][[1, 1]]
N@%
```

Out[•]= $x \to \frac{1}{3}\left(1 + \left(19 - 3\sqrt{33}\right)^{1/3} + \left(19 + 3\sqrt{33}\right)^{1/3}\right)$

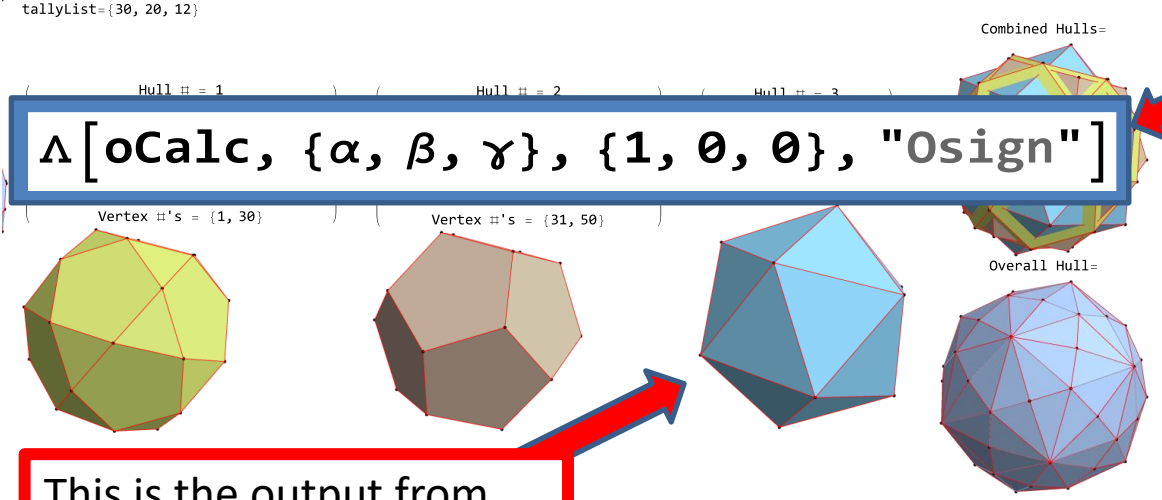
Out[•]= $x \to 1.83929$

Generating Solid Convex Hulls from Quaternions and their Weyl Group Orbits

```
λ1 = (τ + 3)/5;
η1 = (2 σ + 3)/3;
q1 = (τ e1 - 2 σ e2 + τ^3 e3) / 2 /. slRep;
q1p = (2 e1 + √5 e2 + (τ + 2) e3) / 2 /. slRep;
A1 = λ1 (-σ e2 + e3) /. slRep;
B1 = η1 (-σ e1 + τ e3) /. slRep;
C1 = (e1 - σ e2 + τ e3) / 2 /. slRep;

iNorm = 1 / FullSimplify[Norm@{0, 1, 1 / φ} /. slRep, Assumptions → φωAssumptions];
aNorm = FullSimplify@octNorm[A1 /. slRep];
bNorm = FullSimplify@octNorm[B1 /. slRep];
cNorm = FullSimplify@octNorm[C1 /. slRep];

DisdyakisTriacontahedron = FullSimplify[Join[{
    (* Unit scaled Icosahedron (12) **)
    {"H3", 2 iNorm {0, 0, 1}},
    (* Dodecahedron (20=Icosahedron (12) + Cube (8)) **)
    {"H3", bNorm/aNorm 2/√3 {σ, 0, 0}} /. slRep,
    {"H3", bNorm/aNorm 2/√3 {1, -1, 1}, "None"},
    (* Icosidodecahedron (30=Octahedron (6)+Rhombicuboctahedron (24)) **)
    {"H3", 1/aNorm {0, σ, 0}},
    {"H3", 1/aNorm {-1, τ, σ}}}] /. slRep, Assumptions → φωAssumptions];
hulls3D@oScaleListDo@%
tallyList={30, 20, 12}
```




```
Λ[oCalc, {α, β, γ}, {1, 0, 0}, "Osign"]
```

This is the output from the example code on the left.

This is example code that initiates the generation of the output.

hulls3D gathers the vertices by their 3D norm and calculates each concentric convex hull with increasing opacity and varying colors.

The **Λ** function takes as input a function (e.g. **oCalc** to only calculate the orbit vertices, vs. **oShow** which also generates the convex hulls using hulls3D). Scaling can be applied before or after calling **Λ** or **oCalc**.

In this case, the **{α,β,γ}** are quaternions $\{e_1,e_2,e_3\}$ or the group (e.g. **"H3"**) and may depend on identifying a particular (optional or even scaled) Weyl orbit (i.e. **{1,0,0}** as the **"Parent"** of the diagram). Shown are the $7=2^{\text{rank}=3}-1$ combinations of empty and filled nodes from the 3 node Coxeter-Dynkin diagrams, with **{0,0,0}** being a **"Snub"** (omitted here) and **{1,1,1}** being **"OmniTruncated"**) .

The optional parameter "**Osign**" identifies one of 14 combinations of sign and/or position permutations (i.e. this one takes only Odd sign permutations from the cyclic position permutations (vs. all sign permutations given by the default "**Rotate**").

The 5 Platonic Solids, which includes their Duals

A_3 B_3 H_3

Λ(100) & Λ(001) Tetrahedrons (Self Dual)

Tetrahedron (Self Dual)

010

001

101

110

011

111

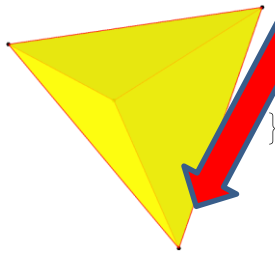
Out[•]= $\frac{2}{\sqrt{3}}$

Out[•]= 1.1547

In[•]:= Λ[oCalc, {α, β, γ}, {1, 0, 0}, "Osign"]
Λ[oShow, {α, β, γ}, {1, 0, 0}, "Osign"]

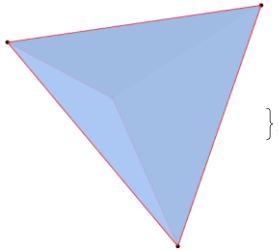
Out[•]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right\}$

Out[•]= { Hull # = 1
with 4 vertices
of 3D Norm = $\frac{\sqrt{3}}{2}$
= $\frac{\sqrt{3}}{2}$
= 0.866
Vertex #'s = {1, 4} ,

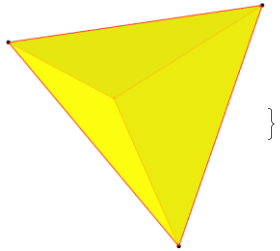


}

{{Overall Hull=,



}, Combined Hulls=,



}

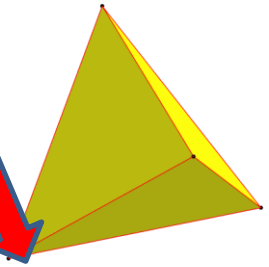
Out[•]= $\frac{2}{\sqrt{3}}$

Out[•]= 1.1547

In[•]:= Λ[oCalc, {α, β, γ}, {0, 0, 1}, "Osign"]
Λ[oShow, {α, β, γ}, {0, 0, 1}, "Osign"]

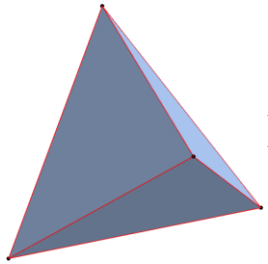
Out[•]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\}, \left\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}\right\}$

Out[•]= { Hull # = 1
with 4 vertices
of 3D Norm = $\frac{\sqrt{3}}{2}$
= $\frac{\sqrt{3}}{2}$
= 0.866
Vertex #'s = {1, 4} ,

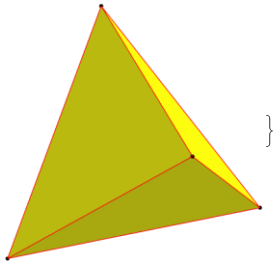


}

{{Overall Hull=,



}, Combined Hulls=,



}

The 5 Platonic Solids, which includes their Duals

A_3 B_3 H_3

Λ(010) Octahedron (see Platonic Dual W(B3) Λ(001) Cube below)

In[•]:= (* Platonic Dual Cube *)

Octahedron (Cube)

010

001

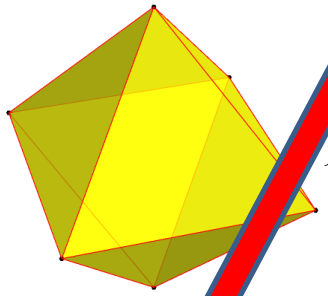
101

110

011

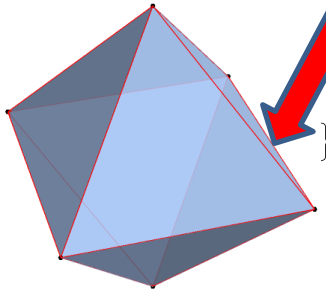
111

Out[•]= { Hull # = 1
with 6 vertices
of 3D Norm = 1
= 1
= 1.
Vertex #'s = {1, 6} ,

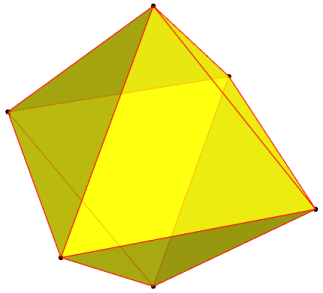


}

{{Overall Hull=,



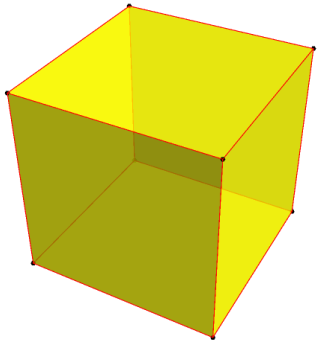
}, Combined Hulls=,



(* Λ(0 Tetrahedron from above *)

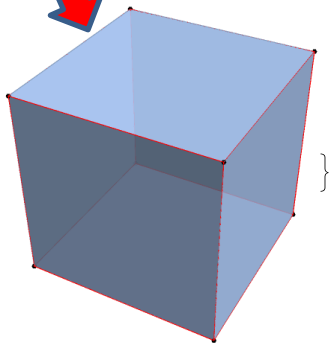
Λ[oCalc, α, β, γ}, {0, 0, 1}, "Osign"]]][[1, 2, 1]]

Out[•]= {{ Hull = 1
with 8 vertices
of 3D Norm $\frac{\sqrt{3}}{2}$
= $\frac{\sqrt{3}}{2}$
= .866
Vertex #'s = {1, ,

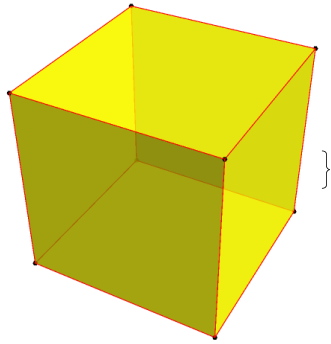


}

, {{Overall Hull=,



}, Combined Hulls=,



}

The 5 Platonic Solids, which includes their Duals

A_3 B_3 H_3

Λ(100) Dodecahedron & Λ(001) Icosahedron (Platonic Solid Duals of each other)

Dodecahedron (Icosahedron)

010

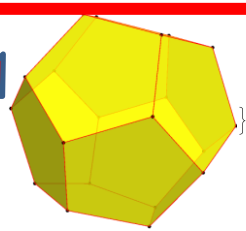
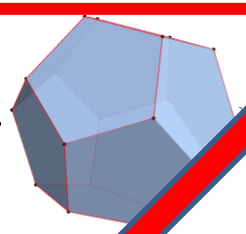
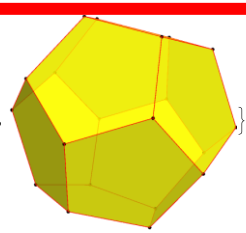
001

101

110

011

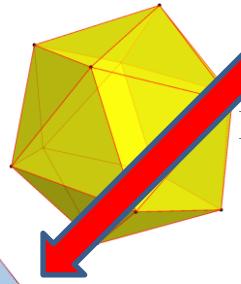
111

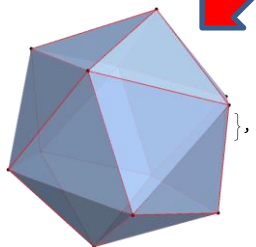
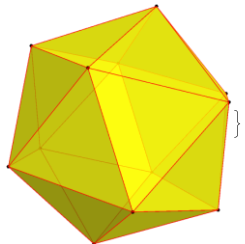
Out[∘]= {{Hull # = 1 with 20 vertices of 3D Norm = $\frac{\sqrt{3}}{2}$ = $\frac{\sqrt{3}}{2}$ = 0.866 Vertex #'s = {1, 20},  }, {{Overall Hull=,  , Combined Hulls=,  }}

In[∘]:= (* Λ(001) Icosahedron *)

In[∘]:= Λ[oCalc, {α, β, γ}, {0, 0, 1}, "Rotate"]
Λ[oShow, {α, β, γ}, {0, 0, 1}, "Rotate"]

Out[∘]= $\left\{\left\{-\frac{1}{2}, -\frac{1}{2\varphi}, 0\right\}, \left\{-\frac{1}{2}, \frac{1}{2\varphi}, 0\right\}, \left\{0, -\frac{1}{2}, -\frac{1}{2\varphi}\right\}, \left\{0, -\frac{1}{2}, \frac{1}{2\varphi}\right\}, \left\{0, \frac{1}{2}, -\frac{1}{2\varphi}\right\}, \left\{0, \frac{1}{2}, \frac{1}{2\varphi}\right\}, \left\{\frac{1}{2}, -\frac{1}{2\varphi}, 0\right\}, \left\{\frac{1}{2}, \frac{1}{2\varphi}, 0\right\}, \left\{-\frac{1}{2\varphi}, 0, -\frac{1}{2}\right\}, \left\{-\frac{1}{2\varphi}, 0, \frac{1}{2}\right\}, \left\{\frac{1}{2\varphi}, 0, -\frac{1}{2}\right\}, \left\{\frac{1}{2\varphi}, 0, \frac{1}{2}\right\}\right\}$

Out[∘]= {Hull # = 1 with 12 vertices of 3D Norm = $\frac{1}{2}\sqrt{1+\frac{1}{\varphi^2}}$ = $\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$ = 0.9511 Vertex #'s = {1, 12},  }

{{Overall Hull=,  , Combined Hulls=,  }}

The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Truncated Tetrahedron (Triakis Tetrahedron)

Mirrored Pairs - Irregular Small Rhombicuboctahedron

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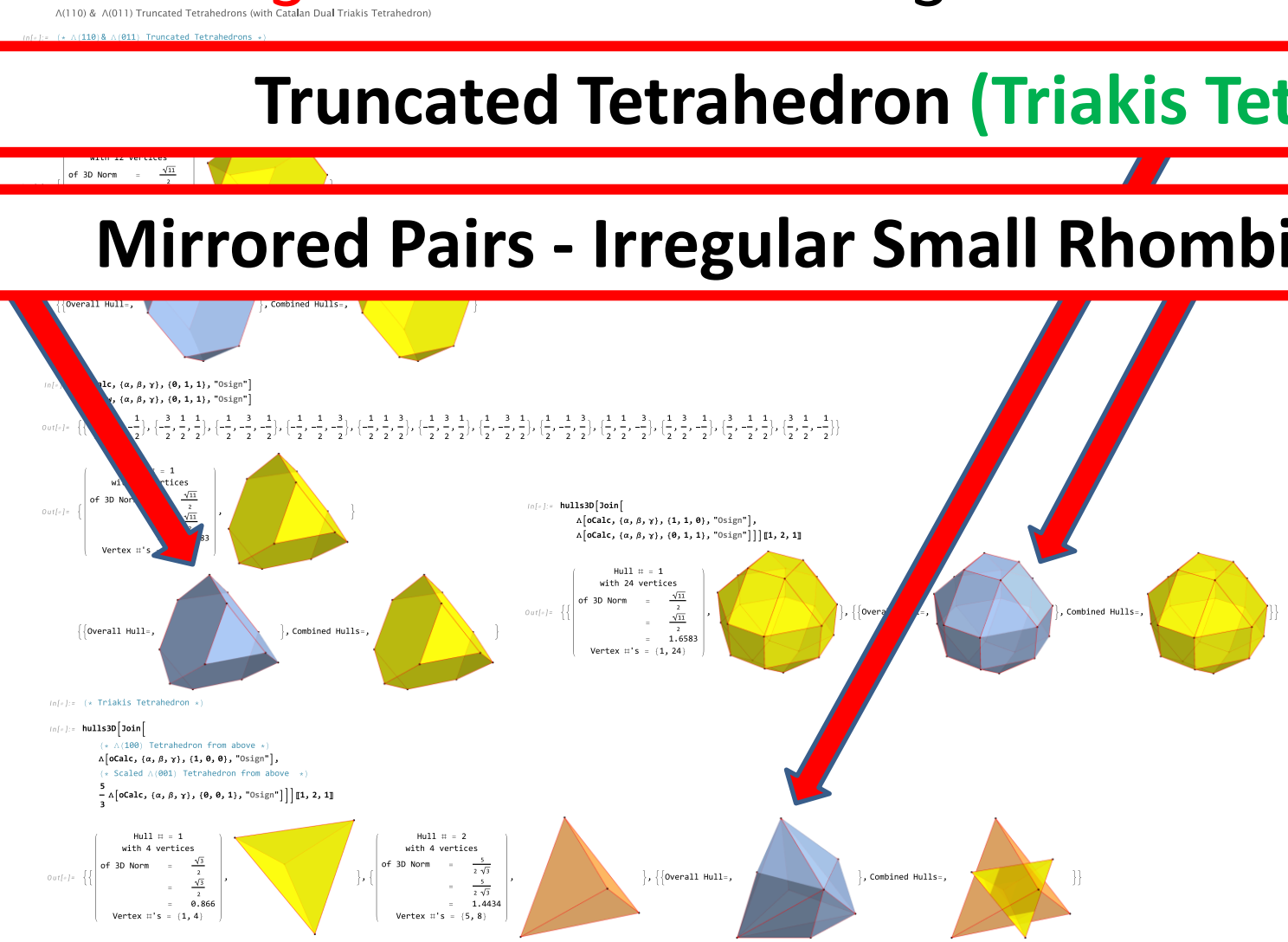

The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Λ(101) Cuboctahedron 3D T' (with Catalan Dual 3D T Rhombic Dodecahedron=Octahedron+Cube)

In[∘]:= hulls3D[Join[

Λ[oCalc, {α, β, γ}, {0, 1, 0}, "Osign"]

Cuboctahedron (Rhombic Dodecahedron)

Λ[oShow, {α, β, γ}, {1, 0, 1}, "Osign"]

Out[∘]= {{−1, −1, 0}, {−1, 0, −1}, {−1, 0, 1}, {−1, 1, 0}, {0, −1, −1},
{0, −1, 1}, {0, 1, −1}, {0, 1, 1}, {1, −1, 0}, {1, 0, −1}, {1, 0, 1}, {1, 1, 0}}

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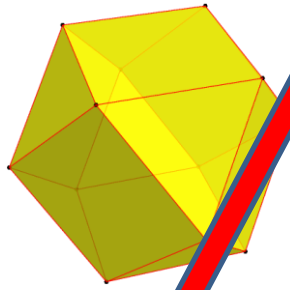
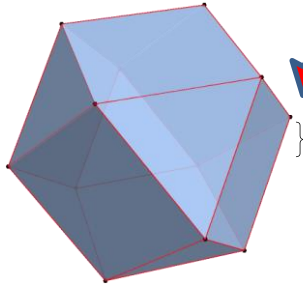
Out[∘]= { Hull # = 1
with 12 vertices
of 3D Norm = $\sqrt{2}$
= $\sqrt{2}$
= 1.4142
Vertex #'s = {1, 12} ,

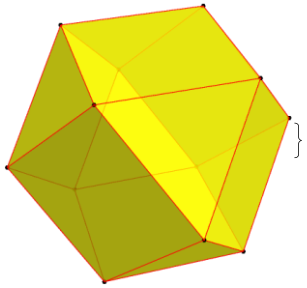


}

{{Overall Hull=,

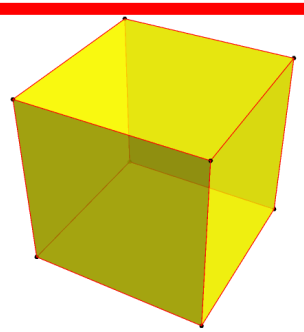


}, Combined Hulls=,

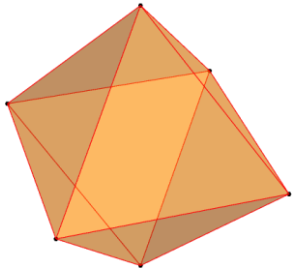


}

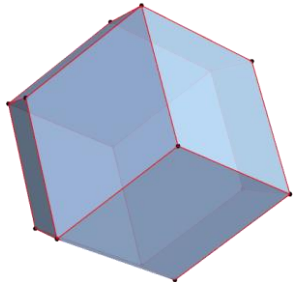
Hull # = 1
with 8 vertices
of 3D Norm = $\frac{\sqrt{3}}{2}$
= $\frac{\sqrt{3}}{2}$
= 0.866
Vertex #'s = {1, 8}



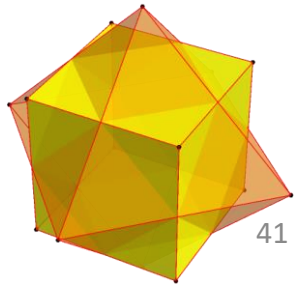
Hull # = 2
with 6 vertices
of 3D Norm = 1
= 1
= 1.
Vertex #'s = {9, 14}



Overall Hull=



Combined Hulls=



The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Truncated Octahedron (Tetrakis Hexahedron)

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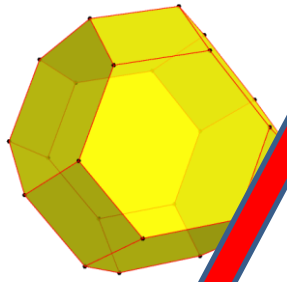
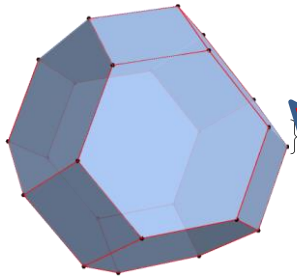
Out[•]= {{-2, -1, 0}, {-2, 0, -1}, {-2, 0, 1}, {-2, 1, 0}, {-1, -2, 0}, {-1, 0, -2}, {-1, 0, 2},
{0, -1, -2}, {0, -1, 2}, {0, 1, -2}, {0, 1, 2}, {0, 2, -1}, {0, 2, 1}, {1, -2, 0},
{-1, 2, 0}, {0, -2, -1}, {0, -2, 1},
{1, 0, -2}, {1, 0, 2}, {1, 2, 0}, {2, -1, 0}, {2, 0, -1}, {2, 0, 1}, {2, 1, 0}}

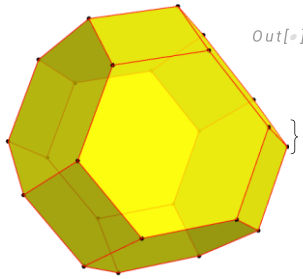
Out[•]= { Hull # = 1
with 24 vertices
of 3D Norm = $\sqrt{5}$
= $\sqrt{5}$
= 2.2361
Vertex #'s = {1, 24} }



{{Overall Hull=,



}, Combined Hulls=,



}}

(* Scaled Δ(010) Octahedron *)

$\frac{\sqrt{3}}{2}$ oCalc, {α, β, γ}, {0, 1, 0}, "Osign"],

(* Cu *)

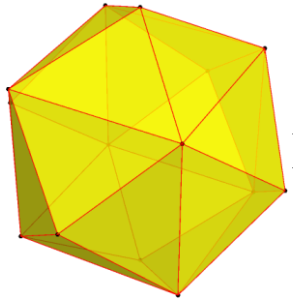
(* Δ(1 Tetrahedron from above *)

Λ[oCalc α, β, γ}, {1, 0, 0}, "Osign"],

(* Δ(001 Tetrahedron from above *)

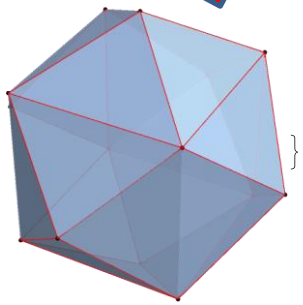
Λ[oCalc, β, γ}, {0, 0, 1}, "Osign"]]][[1, 2, 1]]

Out[•]= {{ Hull = 1
with 14 tices
of 3D Norm $\frac{\sqrt{3}}{2}$
$\frac{\sqrt{3}}{2}$
= 0.866
Vertex #'s = {14} }

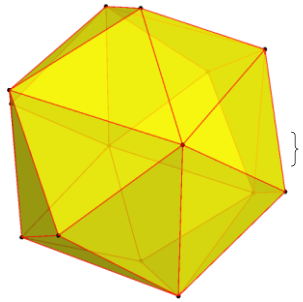


},

{{Overall Hull=,



}, Combined Hulls=,



}}

The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Λ(101) Small Rhombicuboctahedron (with Catalan Dual Deltoidal Icositetrahedron

Small Rhombicuboctahedron (Deltoidal Icositetrahedron)

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Out[∘]= $\left\{\left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}\right\}, \ldots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right\},$

$\left\{-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right\}, \ldots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}\right\},$

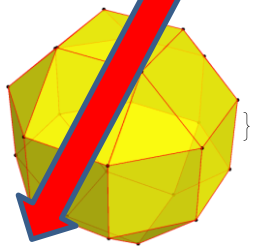
$\left\{-1-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}, \left\{-1-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{-1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ldots\right\}, \left\{-1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\},$

$\left\{-\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}, 1+\ldots, -\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\},$

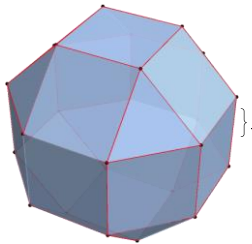
$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, \ldots\right\}, \left\{\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\},$

$\left\{1+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}, \left\{1+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{1+\ldots, -\frac{1}{\sqrt{2}}\right\}, \left\{1+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\}$

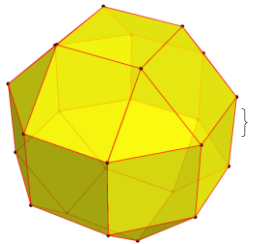
Out[∘]= { Hull # = 1
with 24 vertices
of 3D Norm = $\sqrt{\frac{5}{2}+\sqrt{2}}$
= $\sqrt{\frac{5}{2}+\sqrt{2}}$
= 1.9784
Vertex #'s = {1, 24} }



{{Overall Hull=,



}, Combined Hulls=,



}

In[∘]:= (* Catalan Dual of Λ(101) Small Rhombicuboctahedron (Deltoidal Icositetrahedron) *)

In[∘]:= hulls3D[Join[

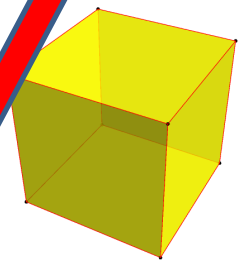
$\sqrt{2}$

(* Scaled Λ(001) Cube from above *)

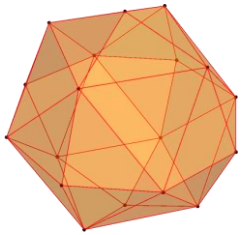
$\sqrt{2}\,\frac{1+\sqrt{8}}{7}$ Λ[oCalc, {α, β, γ}, {0, 0, 1}]]]

Out[∘]= tallyList={8, 18}

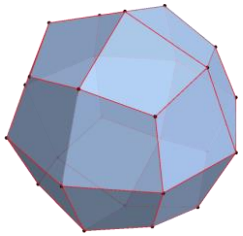
Hull # = 1
with 8 vertices
of 3D Norm = $\frac{1}{7}\left(\sqrt{3}+2\sqrt{6}\right)$
= $\frac{1}{7}\left(\sqrt{3}+2\sqrt{6}\right)$
= 0.9473
Vertex #'s = {1, 8}



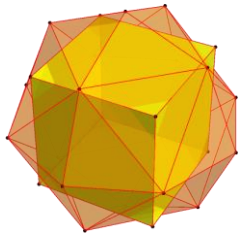
Hull # = 2
with 18 vertices
of 3D Norm = 1
=
=
Vertex #'s = {9



Overall Hull=



Combined Hulls=



The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Truncated Cube (Triakis Octahedron)

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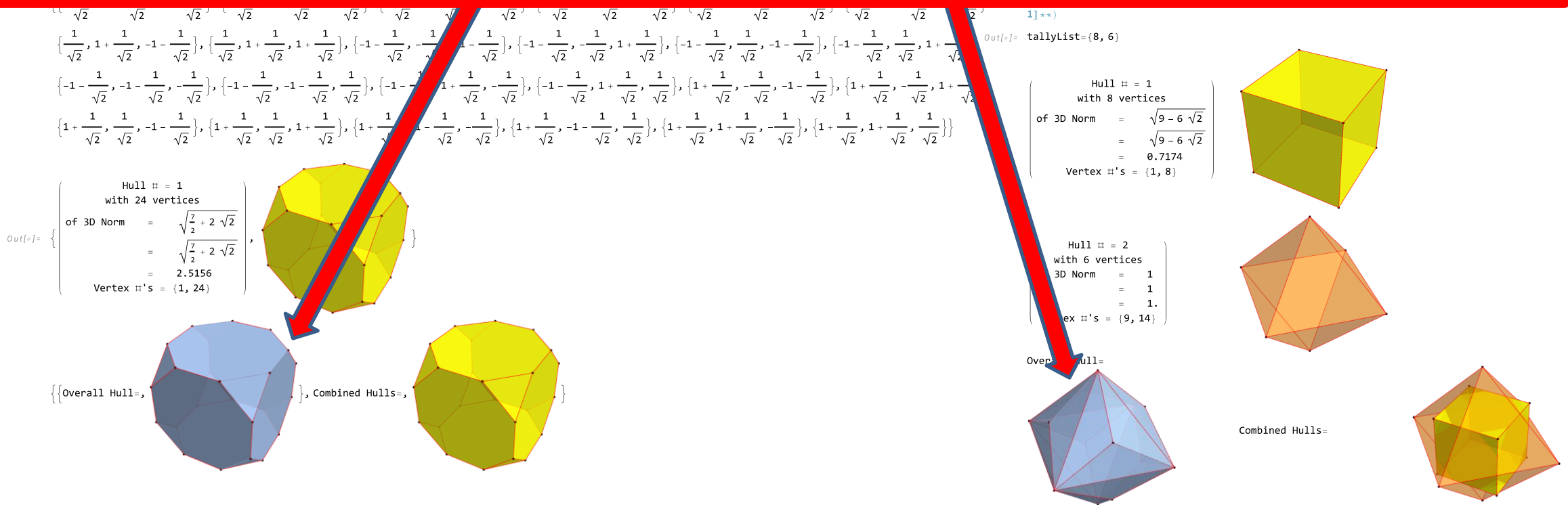

The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Λ(111) Great Rhombicuboctahedron (aka Truncated cuboctahedron) (with Catalan Dual Disdyakis Dodecahedron

In[∘]:= hulls3D[Join[

(* Scaled Δ(100) Octahedron from above *)

Great Rhombicosidodecahedron (Disdyakis Dodecahedron)

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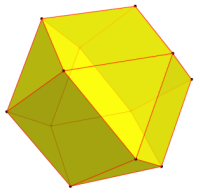
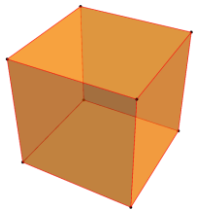
$\left(1+2\sqrt{2}\right)\frac{\sqrt{2/3}}{\sqrt{9+6\sqrt{2}}}\,\Lambda[\text{oCalc},\{\alpha,\beta,\gamma\},\{0,0,1\}]]]$ (*)[[1,2,1]]**)

Out[∘]= tallyList={12, 8, 6}

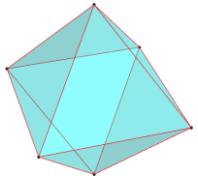
Hull # = 1
with 12 vertices
of 3D Norm = $\frac{1}{7}\left(-1+5\sqrt{2}\right)$
= $\frac{1}{7}\left(-1+5\sqrt{2}\right)$
= 0.8673
Vertex #'s = {1, 12}

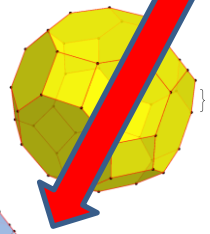


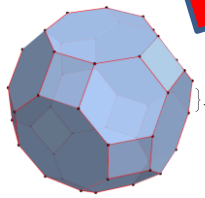
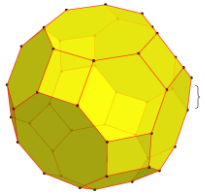
Hull # = 2
with 8 vertices
of 3D Norm = $\sqrt{\frac{11}{3}-2\sqrt{2}}$
= $\sqrt{\frac{11}{3}-2\sqrt{2}}$
= 0.9156
Vertex #'s = {13, 20}



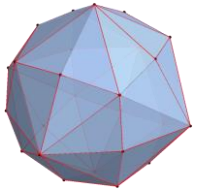
Hull # = 3
with 6 vertices
of 3D Norm = 1
= 1
= 1.
Vertex #'s = {21, 26}



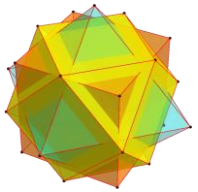
Out[∘]= {Hull # = 1
with 48 vertices
of 3D Norm = $\sqrt{\frac{13}{2}+3\sqrt{2}}$
= $\sqrt{\frac{13}{2}+3\sqrt{2}}$
= 3.2776
Vertex #'s = {1, 48},  }

{{Overall Hull=,  }, Combined Hulls=,  }}

Overall Hull=



Combined Hulls=



The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Chiral Archimedean Snub Cubes (when paired give a non-chiral W(B3) Λ(111) Great Rhombicuboctahedron (aka. Truncated cuboctahedron albeit irregular)

Snub Cube (Pentagonal Icositetrahedron)

Mirrored Pairs - Irregular Great Rhombicosidodecahedron (prev slide)

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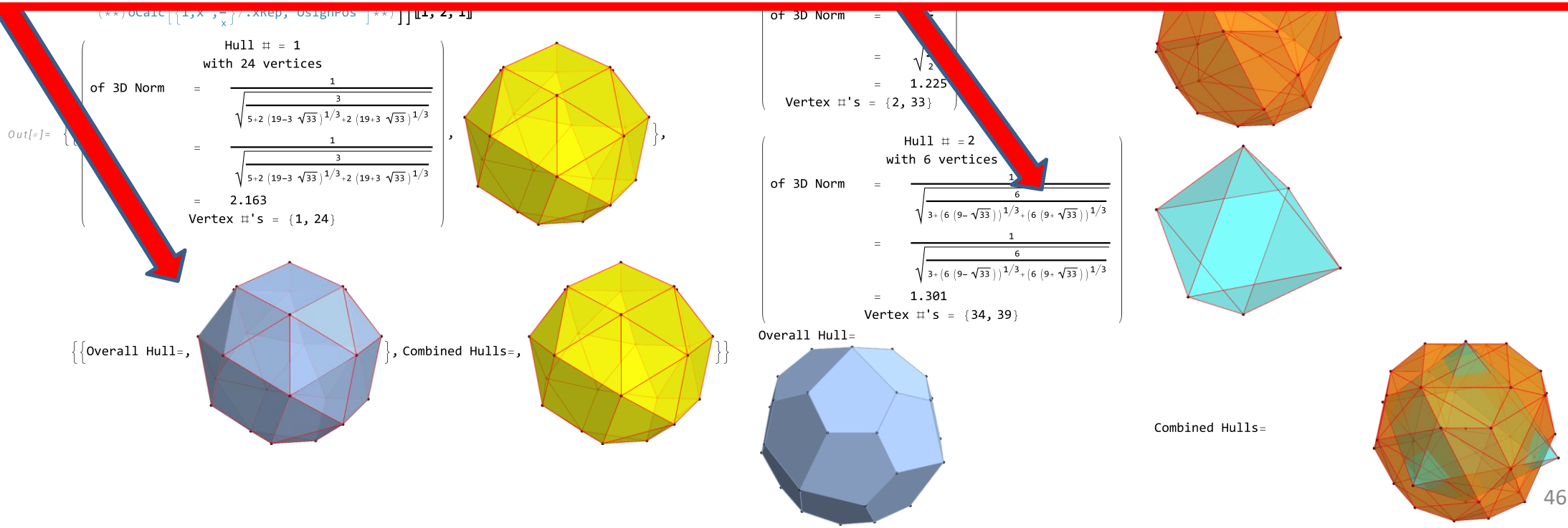

The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Λ(110) Truncated Dodecahedron (with Catalan Dual Triakis Icosahedron)

Truncated Dodecahedron (Triakis Icosahedron)

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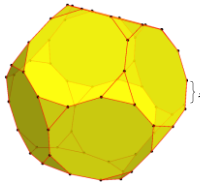
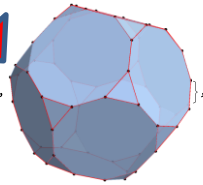
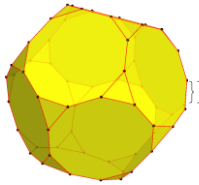
Λ[oCalc, q2, {1, 1, 1}, "Rotate"],
Λ[oCalc, q3b, {1, 1, 1}, "Rotate"]] (**) / Λ110UnitNorm (**  , 1]]
```

Out[=]= {{ Hull # = 1 with 60 vertices of 3D Norm = 1 = 1 = 1. Vertex #'s = {1, 60} , }, {{Overall Hull=, }, Combined Hulls=, }}

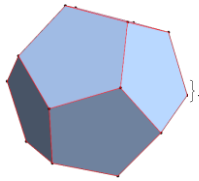




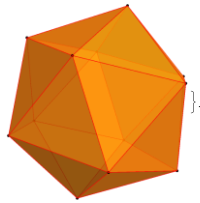


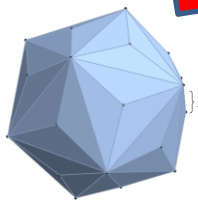
In[=]:=
```
hulls3D[Join[
  (* From the Rhombic Triacontahedron **)
  (* Scaled Icosahedron from Λ(100) & Λ(001) above **)
  Λ[oCalc, {α, β, γ}, {0, 0, 1}, "Rotate"], (**)
  (* Scaled Dodecahedron from Λ(100) & Λ(001) above **)
  1/11 Sqrt[25 + 2 Sqrt[5]] Sqrt[1 + 1/φ^2] (**) Λ[oCalc, σ {α, β, γ}, {1, 0, 0}, "Rotate"],
  1/11 Sqrt[25 + 2 Sqrt[5]] Sqrt[1 + 1/φ^2] (**) Λ[oCalc, {α, β, γ}, {1, -1, 1}, False]]][[1, 2, 1]]
```

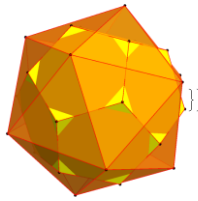
Out[=]= {{ Hull # = 1 with 20 vertices of 3D Norm = $\frac{1}{22}\sqrt{25+2\sqrt{5}}\sqrt{1+\frac{1}{\varphi^4}+\frac{1}{\varphi^2}+\varphi^2}$ = $\frac{1}{22}\sqrt{\frac{15}{2}\left(27+7\sqrt{5}\right)}$ = 0.813 Vertex #'s = {1, 20} , }, }



{ Hull # = 2 with 12 vertices of 3D Norm = $\frac{1}{2}\sqrt{1+\frac{1}{\varphi^2}}$ = $\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$ = 0.9511 Vertex #'s = {21, 32} , }, {{Overall Hull=, }, Combined Hulls=, }}







The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Λ(010) Icosidodecahedron (with Catalan Dual Rhombic Triacontahedron)

Icosidodecahedron (Rhombic Triacontahedron)

010

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101

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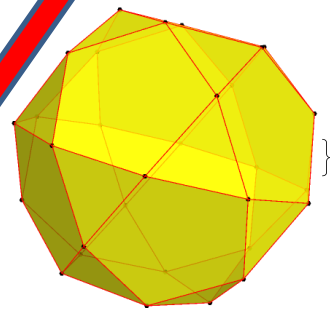
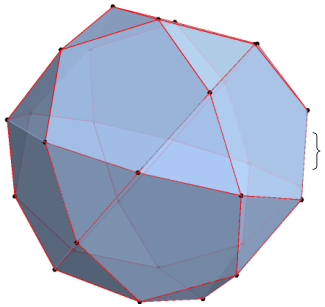
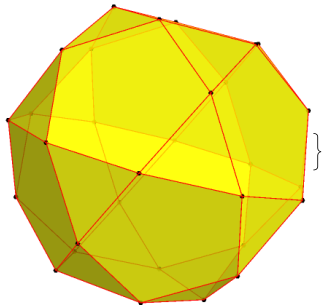
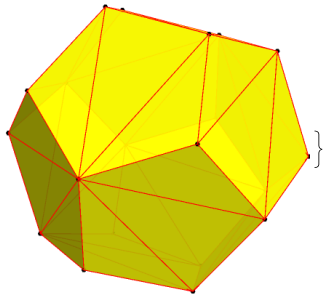
```
Λ[oCalc, σ {α, β, γ}, {0, 1, 0}],
Λ[oCalc, {1, σ, τ} / 2, {1, 1, 1}, "Rotate"]]][[1, 2, 1]]
```

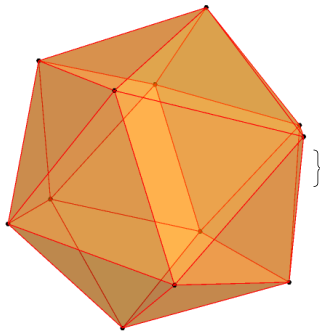
Out[•]= {{ Hull # = 1
with 30 vertices
of 3D Norm = ...
= ...
= 1.
Vertex #'s = {1, 30} }, 

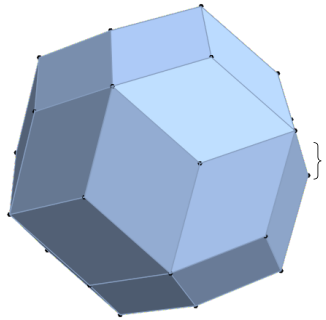
, {{Overall Hull=, 

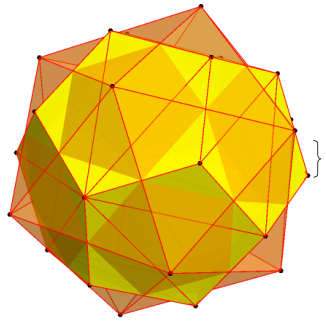
Combined Hulls=, 

```
Λ[oCalc, σ {α, β, γ}, {1, 0, 0}, "Rotate"],
Λ[oCalc, {α, β, γ}, {1, -1, 1}, False],
(* Icosahedron from Λ(100) & Λ(001) above *)
Λ[oCalc, {α, β, γ}, {0, 0, 1}, "Rotate"]]][[1, 2, 1]]
```

Out[•]= {{ Hull # = ...
with 20 vertices
of 3D Norm = $\sqrt{3}$
= ...
= 0.6
Vertex #'s = {1, 20} }, 

}, { Hull # = 2
with 12 vertices
of 3D Norm = $\frac{1}{2}\sqrt{1+\frac{1}{\varphi^2}}$
= $\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$
= 0.951
Vertex #'s = {21, 32} }, 

{{Overall Hull=, 

}, Combined Hulls=, 

The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Truncated Icosahedron (Pentakis Dodecahedron)

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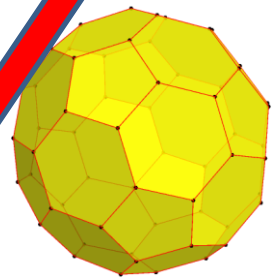
Λ(011) Truncated Icosahedron (with Catalan Dual Pentakis Dodecahedron)

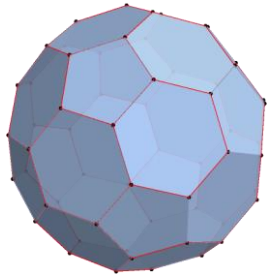
```
Λ[oCalc, τ{2, σ, τ - σ}(*)/2**), {1, 1, "Rotate"],
Λ[oCalc, τ{2σ, τ², 1}(*)/2**), {1, 1, "Rotate"]
](**)/Λ011UnitNorm(**)]
```

Out[•]= tallyList={60}

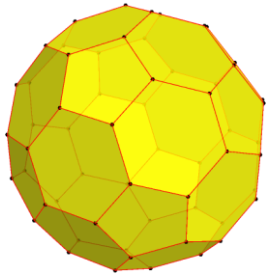
Hull # = 1
with 60 vertices
of 3D Norm = 1
= 1
= 1
Vertex #'s = {1, 60}



Overall Hull=



Combined Hulls=

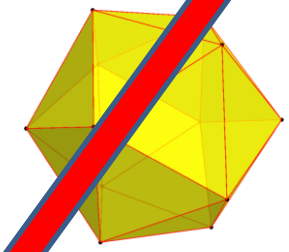


In[•]:= hulls3D[Join[

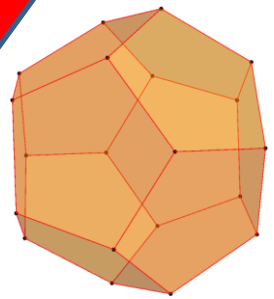
]]

Out[•]= tallyList={12, 20}

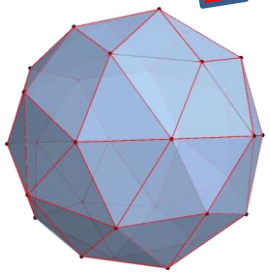
Hull # = 1
with 12 vertices
of 3D Norm = $\sqrt{\frac{3+\frac{3}{\varphi^2}}{(-4+\varphi)^2}}$
= $\frac{1}{19}\sqrt{195+66\sqrt{5}}$
= 0.9742
Vertex #'s = {1, 12}



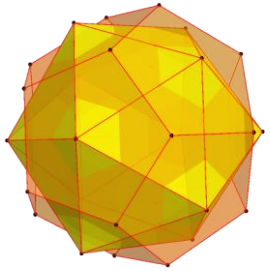
Hull # = 2
with 20 vertices
of 3D Norm = 1
= 1
= 1.
Vertex #'s = {13, 32}



Overall Hull=



Combined Hulls=



The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

Λ(101) Small Rhombicosidodecahedron (with Catalan Dual Deltoidal Hexacontahedron

Small Rhombicosidodecahedron (Deltoidal Hexacontahedron)

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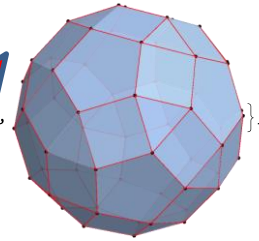
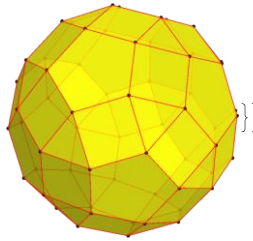
```
Λ[oCalc, {τ - σ, τ, 0} / 2, {1, 1, 1}, "Rotate"],
Λ[oCalc, {2, 1, τ} / 2, {1, 1, 1}, "Rotate"]]][[1, 2, 1]]
```

Hull # = 1
with 60 vertices
Vertex #'s = {1, 60}

{{Overall Hull=,



}, Combined Hulls=,

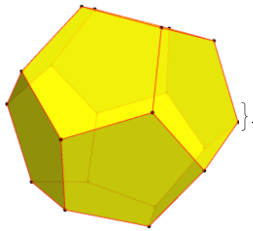


}}

```
In[#]:= (* Catalan Dual of Λ((101) Small Rhombicosidodecahedron (Deltoidal Hexacontahedron) *)
In[#]:= hulls3D[Join[
          (* Rhombic Triacontahedron **)
          (* Scaled Icosahedron from Λ(001) above **)
          Λ[oCalc, {α, β, γ}, {0, 0, 1}, "Rotate"], (**)
          (* Scaled Dodecahedron from Λ(100) above **)
          3/11 Sqrt[5 - 2/Sqrt[5]] Sqrt[1 + 1/φ^2] Λ[oCalc, σ {α, β, γ}, {1, 0, 0}, "Rotate"],
          3/11 Sqrt[5 - 2/Sqrt[5]] Sqrt[1 + 1/φ^2] Λ[oCalc, {α, β, γ}, {1, -1, 1}, False], (**)
          (* Scaled Λ(010) Icosidodecahedron (30) from above **)
          (* Octahedron **)
          3/2 Sqrt[1 - 2/Sqrt[5]] Sqrt[1 + 1/φ^2] Λ[oCalc, σ {α, β, γ}, {0, 1, 0}],
          3/2 Sqrt[1 - 2/Sqrt[5]] Sqrt[1 + 1/φ^2] Λ[oCalc, {1, σ, τ} / 2, {1, 1, 1}, "Rotate"] (**)]][[1, 2, 1]]
```

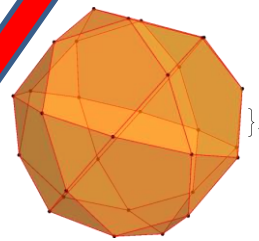
Out[#]= {{

Hull # = 1
with 20 vertices
of 3D Norm $= \frac{3}{22}\sqrt{5-\frac{2}{\sqrt{5}}}\sqrt{1+\frac{1}{\varphi^4}+\frac{1}{\varphi^2}+\varphi^2}$
$= \frac{3}{22}\sqrt{\frac{3}{2}\left(23+3\sqrt{5}\right)}$
$= 0.9103$
Vertex #'s = {1, 20}



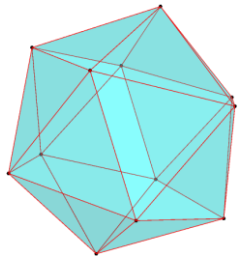
}, {

Hull # = 2
with 30 vertices
of 3D Norm $= \frac{3\sqrt{1+\frac{1}{\varphi^2}}\sqrt{-1+\ldots}}{2\cdot 5^{1/4}\ldots}$
$= \frac{3}{4}\left(-1+\ldots\right)$
$= 0.92\ldots$
Vertex #'s = {2…, …}

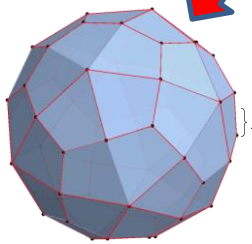


},

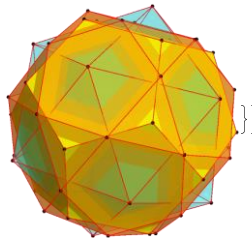
Hull # = 3
with 12 vertices
of 3D Norm $= \frac{1}{2}\sqrt{1+\frac{1}{\varphi^2}}$
$= \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$
$= 0.9511$
Vertex #'s = {51, 62}



}, {{Overall Hull=,



}, Combined Hulls=,



}}

The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 | B_3 | H_3

Chiral Archimedean Snub Dodecahedron (aka. snub icosidodecahedron)

Catalan Pentagonal Hexacontahedron

Out[•]= tallyList={80, 12}

Snub Dodecahedron (Pentagonal Hexacontahedron)

Mirrored Pairs - Irregular Great Rhombicosidodecahedron (next slide)

In[•]:= -oCalc[{2, τ - σ, τ + 2} / 2, "OsignPos"],

hulls3D[Flatten[φτsnubDodecaHalf, 1]]

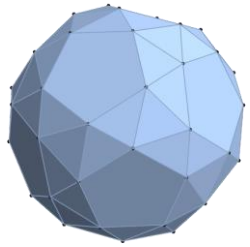
tallyList={60}

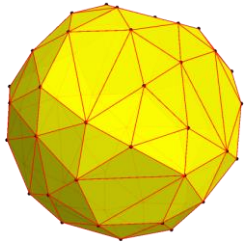
Hull # = 1
with 60 vertices

Norm = $\frac{(-1+\sqrt{5})\sqrt{\left(3+\frac{1}{\varphi}\right)^2+2\varphi^2}}{\sqrt{74-18\sqrt{5}}}$
= 1
= 1.

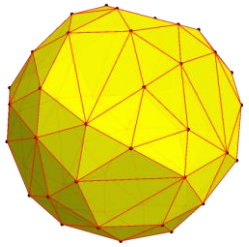
Vertex #'s = {1, 60}

Overall Hull=





Combined Hulls=



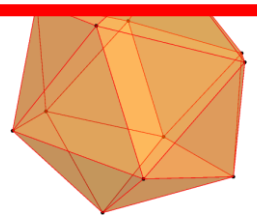
(* Orbit 2 unit scale Icosahedron (12) *)

$\Lambda\left[\text{oCalc}, \frac{\{\alpha, \beta, \gamma\}}{\frac{1}{2}\sqrt{1+\frac{1}{\varphi^2}}}, \{0, 0, 1\}, \text{"OsignPos"}\right]$,

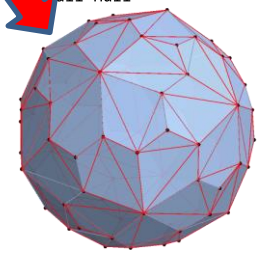
(* Orbit 3 phScale Chiral snub dodecahedron (60) *)

phScale ρc2OctList]]

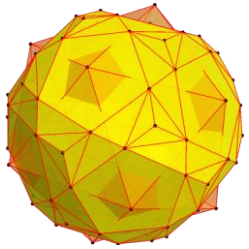
Hull # = 2
with 12 vertices
of 3D Norm = 1
= 1
= 1.
Vertex #'s = {81, 92}



Overall Hull=



Combined Hulls=



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The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

A_3 B_3 H_3

In[∘]:= GreatRhombicosidodecahedron = {

In[∘]:= DisdyakisTriacontahedron = FullSimplify[Join[{

Great Rhombicosidodecahedron (Disdyakis Triacontahedron)

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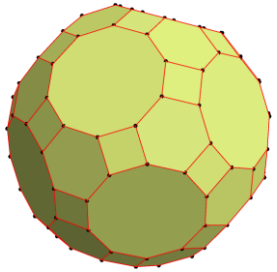
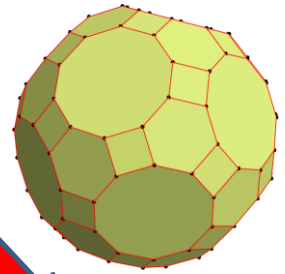
hulls3D@o[illegible]eListDo@%

Out[∘]= tallyList={12[illegible]

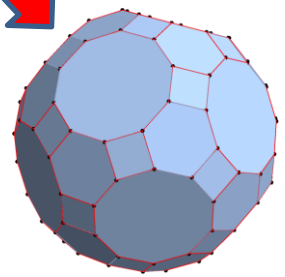
Hull # [illegible]
with 120 ver[illegible]
of 3D Norm = $\frac{1}{2}\sqrt{9+\frac{6}{[illegible]}+2\varphi^2}$
= $\sqrt{\frac{33}{8}+\frac{5\sqrt{5}}{8}}$
= 2.35
Vertex #'s = {1, 120}



Combined Hulls=



Overall Hull=



$\left\{\text{"H3"}, \frac{\text{bNorm}}{\text{aNorm}}\frac{2}{\sqrt{3}}\{1,-1,1\}, \text{"None"}\right\}$,

(* Icosidodecahedron (30=Octahedron (6)+Rhombicubocto[illegible]ron (24)) **)

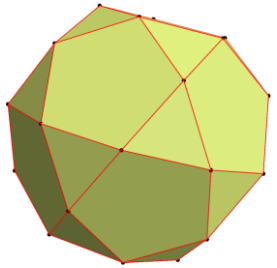
$\left\{\text{"H3"}, \frac{1}{\text{aNorm}}\{0,\sigma,0\}\right\}$,

$\left\{\text{"H3"}, \frac{1}{\text{aNorm}}\{-1,\tau,\sigma\}\right\}\}]$ /. slRep, Assumptions → φωAssumptions]

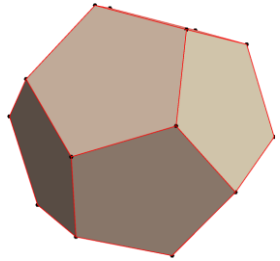
hulls3D@oScaleListDo@%

tallyList={30, 20, 12}

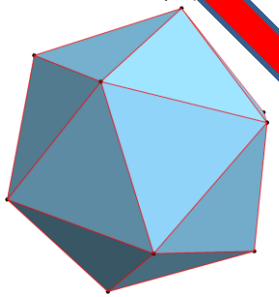
Hull # = 1
with 30 vertices
of 3D Norm = $\frac{5\varphi}{(1+3\varphi)\sqrt{1+\varphi^2}}$
= $\frac{1}{11}\sqrt{5(25-2\sqrt{5})}$
= 0.921
Vertex #'s = {1, 30}



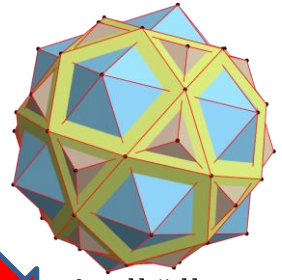
Hull # = 2
with 20 vertices
of 3D Norm = $\frac{5(3-2\varphi)\varphi}{\sqrt{3}(1+3\varphi)\sqrt{1+\varphi^2}}$
= $\sqrt{\frac{5}{3}(5-2\sqrt{5})}$
= 0.938
Vertex #'s = {31, 50}



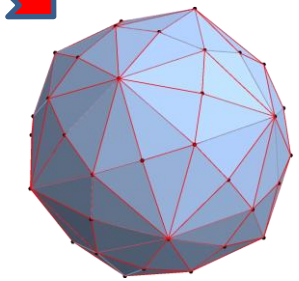
Hull # = 3
with [illegible] vertices
of 3D Nor[illegible] = 1
1
1.
Vertex #'s = {51,



Combined Hulls=



Overall Hull=



The 13 Archimedean Solids with their Catalan Duals with some Johnson and Near-miss Johnsons

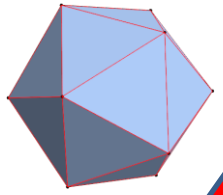
```
In[∘]:= ILΦ = octSimplify@# & /@ Flatten@prq[octPwrαL, TL, 1];
        Length@%
        listILΦ = oct2List[#] & /@ %%;
        hulls3DPerms["listILΦ", False, , 1]

Out[∘]= 120
```

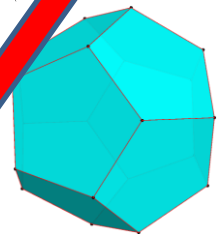
octPwrαL = $\begin{pmatrix} 1 \\ \frac{\varphi}{2}+\frac{e_1}{2}+\frac{e_2}{2\varphi} \\ -\frac{1}{2\varphi}+\frac{\varphi e_1}{2}+\frac{e_2}{2} \\ -\frac{1}{2\varphi}-\frac{\varphi e_1}{2}-\frac{e_2}{2} \\ \frac{\varphi}{2}-\frac{e_1}{2}-\frac{e_2}{2\varphi} \end{pmatrix}$ cpL = $\frac{e_2}{\sqrt{2}}-\frac{e_3}{\sqrt{2}}$

Pentakis Icosidodecahedron (Chamfered Dodecahedron)

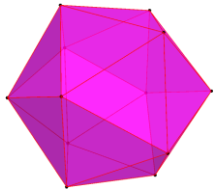
of 3D Norm = $\frac{\sqrt{1+\varphi^2}}{2}$ = $\frac{1}{2}\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}$ = 0.5878
Vertex #'s = {3, 26}



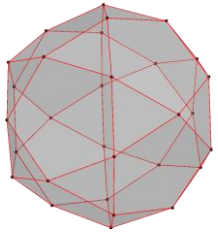
of 3D Norm = $\frac{1}{2}$ = $\frac{\sqrt{3}}{2}$ = 0.866
Vertex #'s = {27, 66}



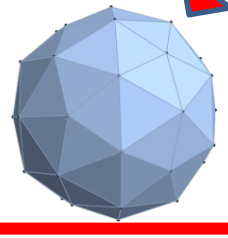
of 3D Norm = $\frac{1}{2}\sqrt{1+\frac{1}{\varphi^2}}$ = $\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$ = 0.9511
Vertex #'s = {67, 90}



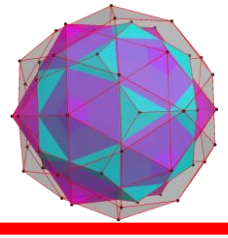
of 3D Norm = 1 = 1 = 1.
Vertex #'s = {91, 120}



Overall Hull=

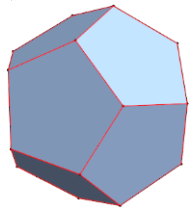


Combined Hulls=

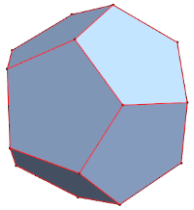


ListName= listJLΦ
Dims used={1, 2, 3}
tallyList={40, 40, 60, 120}
{120, 120, 40, 60}

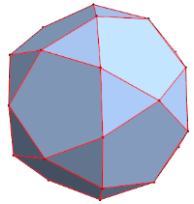
Hull # = 1
with 40 vertices
of 3D Norm = $\frac{1}{2}\sqrt{\frac{3}{2}}\,\varphi$ = $\frac{1}{4}\sqrt{9-3\sqrt{5}}$ = 0.3785
Vertex #'s = {1, 40}



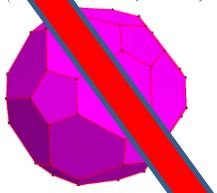
Hull # = 2
with 40 vertices
of 3D Norm = $\frac{\sqrt{\frac{3}{2}}}{2}$ = $\frac{\sqrt{\frac{3}{2}}}{2}$ = 0.6124
Vertex #'s = {41, 80}



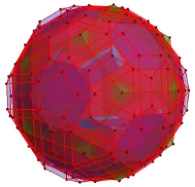
Hull # = 3
with 60 vertices
of 3D Norm = $\frac{1}{\sqrt{2}}$ = $\frac{1}{\sqrt{2}}$ = 0.7071
Vertex #'s = {81, 140}



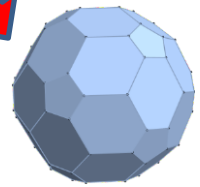
Hull # = 4
with 120 vertices
of 3D Norm = $\sqrt{\frac{1}{4\varphi^2}+\frac{\varphi^4}{8}}$ = $\frac{\sqrt{13-\sqrt{5}}}{4}$ = 0.8202
Vertex #'s = {141, 260}



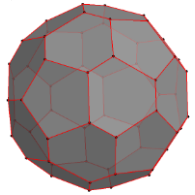
Combined Hulls=



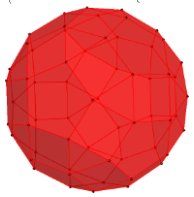
Overall Hull=



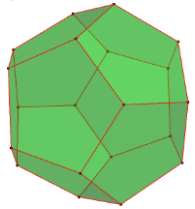
Hull # = 5
with 120 vertices
of 3D Norm = $\frac{\sqrt{\frac{7}{2}}}{2}$ = $\frac{\sqrt{\frac{7}{2}}}{2}$ = 0.9354
Vertex #'s = {261, 380}



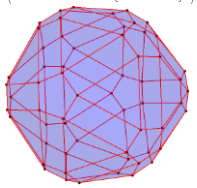
Hull # = 6
with 120 vertices
of 3D Norm = $\sqrt{\frac{1}{8\varphi^4}+\frac{\varphi^2}{4}}$ = $\frac{\sqrt{13-\sqrt{5}}}{4}$ = 0.9758
Vertex #'s = {381, 500}



Hull # = 7
with 40 vertices
of 3D Norm = $\frac{\sqrt{\frac{3}{2}}}{2\varphi}$ = $\frac{1}{8}\left(\sqrt{6}+\sqrt{30}\right)$ = 0.9908
Vertex #'s = {501, 540}



Hull # = 8
with 60 vertices
of 3D Norm = 1 = 1 = 1.
Vertex #'s = {541, 600}



These are 3D (x,y,z) convex hulls projected from 4D group H4 (shown here with 4th Dim w=0)

Agenda

- Introducing the Mathematica™ based VisibLie-E8 Tool
  - The Coxeter-Dynkin Visualizer (Groups, Weyl Orbits, Hasse Diagrams, etc.)
  - The (Bi)Quaternion/(Bi)Octonion/Sedenion Engine (Fano Plane/Cube/Tesseract, etc.)
  - The VisibLie_E8 Hyperdimensional Visualizer (2D/3D Projections, etc.)
- Introducing the Dimensions (0 to 8, ∞)
  - 0D points, 1D edges & 2D faces (A2, B2=C2, G2, H2 Groups)
  - 3D cells & polyhedra (A3, B3, H3 Groups)
  - 4D cells & polychora (A4, C4, D4, F4, H4 Groups)
  - 5D-8D cells & polytopes (A5, C6, D6, E6, E7, E8=BC8+D8 Groups)
- Generating Solid Convex Hulls from Quaternions and their Weyl Group Orbits
  - The 5 Platonic Solids, which includes duals
  - The 13 Archimedean Solids and their Catalan Duals, with some Johnson and Near-miss Johnsons
- 4D Polychora w/Duals, including the Dual Snub 24-Cell

The 600-Cell / 120-Cell Quaternion 3D Structure

(T)etrahedral elements=D4(24)=16-Cell(BC4)+8-Cell(TriRectified BC4)=24-Cell

Alternate (or "Dual" of Self Dual D4) T'=Rectified BC4 (16-Cell):

F4(48)=T+T' and T+T'+ϕ(T+T') are 2X(48 of 112) integer vertex parts of E8 folded to (I=H4)+ϕH4

Orthogonal projection in 3 dimensions with {x=1,y=1,z=1,w=0}

all other permutations simply rotate the structure

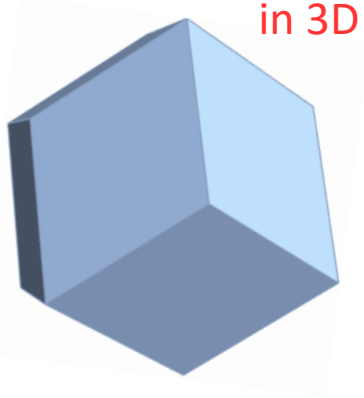
Sorted Positive (2*2+2*4) T=

$$\phi T=\begin{pmatrix} \{\varphi,\ \varphi e_1\} \\ \{\varphi e_2,\ \varphi e_3\} \\ \left\{\frac{1}{2}\varphi\,(1+e_1+e_2+e_3),\ \frac{1}{2}\varphi\,(1+e_1+e_2-e_3),\ \frac{1}{2}\varphi\,(1+e_1-e_2+e_3),\ \frac{1}{2}\varphi\,(1+e_1-e_2-e_3)\right\} \\ \left\{\frac{1}{2}\varphi\,(1-e_1+e_2+e_3),\ \frac{1}{2}\varphi\,(1-e_1+e_2-e_3),\ \frac{1}{2}\varphi\,(1-e_1-e_2+e_3),\ \frac{1}{2}\varphi\,(1-e_1-e_2-e_3)\right\} \end{pmatrix}$$

Catalan Solid

Rhombic Dodecahedron

T 24-Cell in 3D



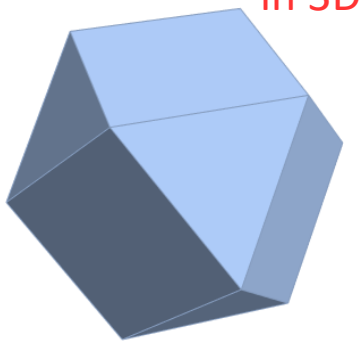
Sorted Positive (2*3+3*2) T'=

$$\phi T'=\begin{pmatrix} \left\{\frac{\varphi\,(1+e_1)}{\sqrt{2}},\ \frac{\varphi\,(1+e_2)}{\sqrt{2}},\ \frac{\varphi\,(1+e_3)}{\sqrt{2}}\right\} \\ \left\{\frac{\varphi\,(1-e_3)}{\sqrt{2}},\ \frac{\varphi\,(1-e_2)}{\sqrt{2}},\ \frac{\varphi\,(1-e_1)}{\sqrt{2}}\right\} \\ \left\{\frac{\varphi\,(e_1+e_2)}{\sqrt{2}},\ \frac{\varphi\,(e_1+e_3)}{\sqrt{2}}\right\} \\ \left\{\frac{\varphi\,(e_1-e_3)}{\sqrt{2}},\ \right. \\ \left\{\frac{\varphi\,(e_2+e_3)}{\sqrt{2}},\ \right. \end{pmatrix}$$

Archimedian Solid

Cuboctahedron

T' 24-Cell in 3D



The 600-Cell / 120-Cell Quaternion 3D Structure

16-Cell+8-Cell=24-Cell Tetrahedral elements:

(T+ϕT) & (T'+ ϕT')

Orthogonal projection in 3 dimensions with {x=1,y=1,z=1,w=0}

all other te the structure

3D Rhombic Dodecahedron (6+8=14 vertex)

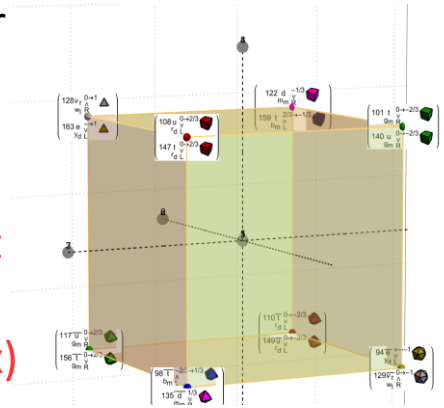
Cube Stellation

vs.

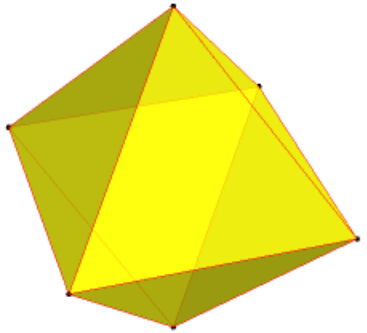
Cube Truncation

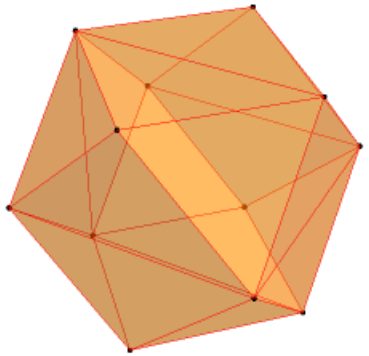
3D Cubocta-Hedron (12 vertex)

3D Cube from 4D Tesseract or 8-Cell (16 vertex)



3D Octahedron + 2@origin from 4D 16-Cell (8 vertex)





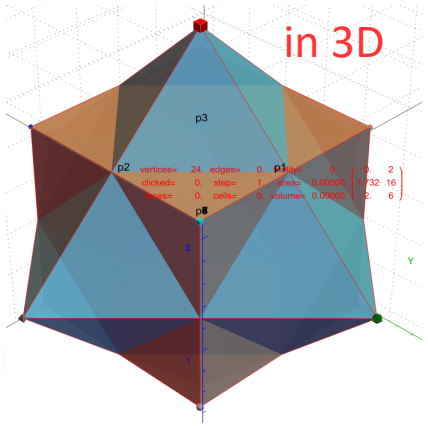
3D Norm r= √3/2

3D Norm r=1

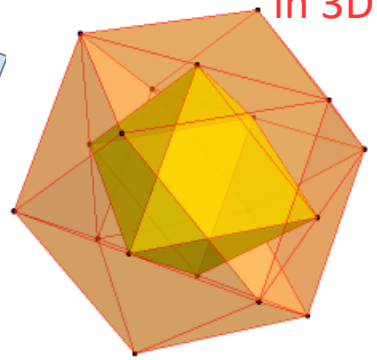
3D Norm r=1/ √ 2

3D Norm r=1

T 24-Cell in 3D



T' 24-Cell in 3D



The 600-Cell / 120-Cell Quaternion Structure

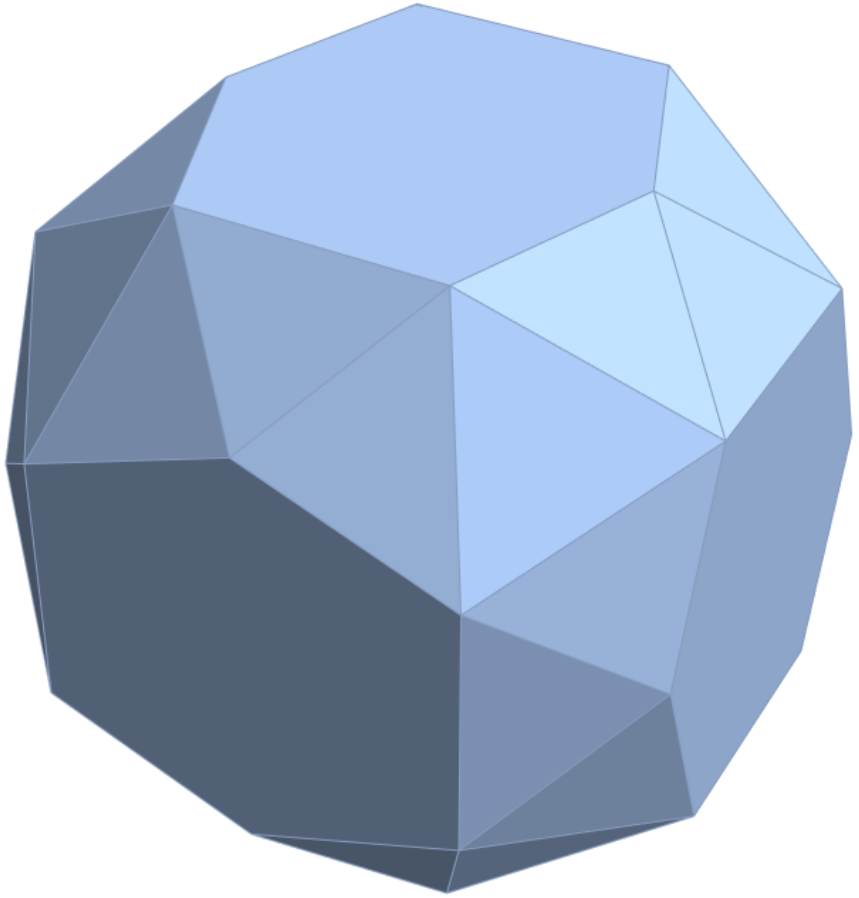
- Both the 120 vertex 600-Cell(120) and the 600 vertex 120-Cell(600) can be generated from quaternion T(24) or T'(24) basis vectors one or two exponents (i,j=0-4) on one or two generator vertices (p^i, p^j) (see Koca 2011 et al.).
- The H4 600-Cell Icosahedral group I(120) (or its alternate I' using T' instead of T) is generated from p^i only.
  - This gives us the ability to map the 120-Cell vertices to each of the 5 copies of 600-Cell vertices.
  - This mapping includes the other orbits of the Weyl group W(D4) as well, e.g. M(192)=M1(96)+M2(96), and N(288).
- As done with the 600-Cell, implementing the 120-Cell in this manner allows the use of not only two the scaled copies of the 120-Cell (J & Jϕ).

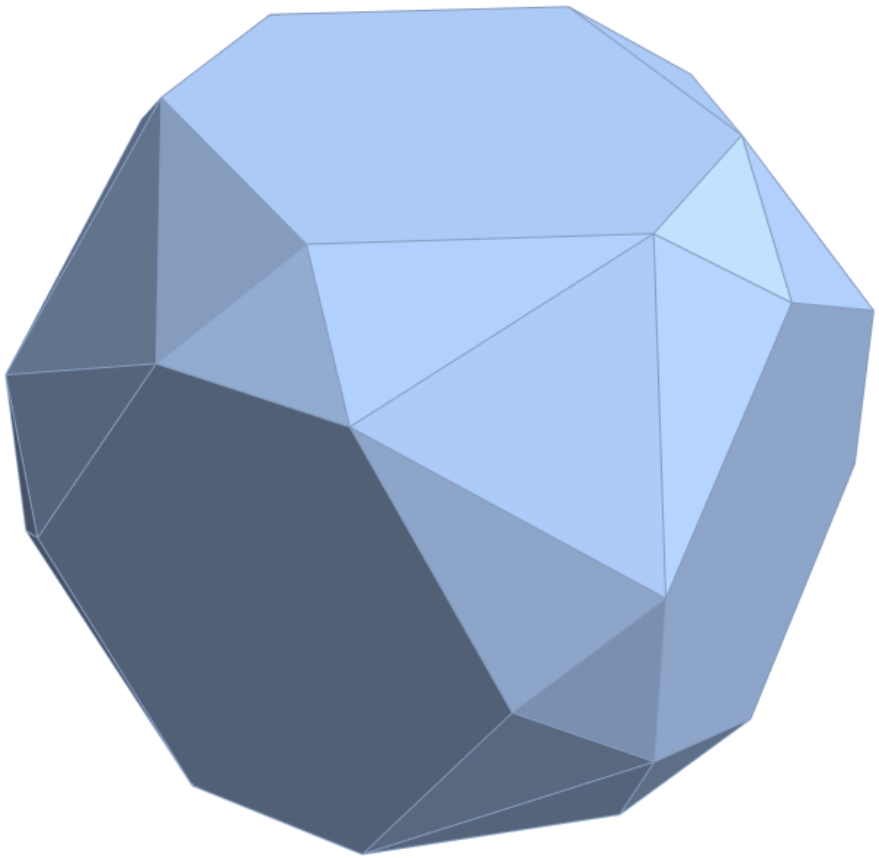
The 600-Cell / 120-Cell Quaternion 3D Structure

Snub 24-Cell (S=I-T) (96) and Alternate (S‘=I’-T’)

$$S = \sum_{i=1}^{4} p^i \circ \left(\overline{p}^{i\dagger} \circ T\right)$$

$$S' = \sum_{i=1}^{4} p^i \circ \left(\overline{p}^{i\dagger} \circ T'\right)$$



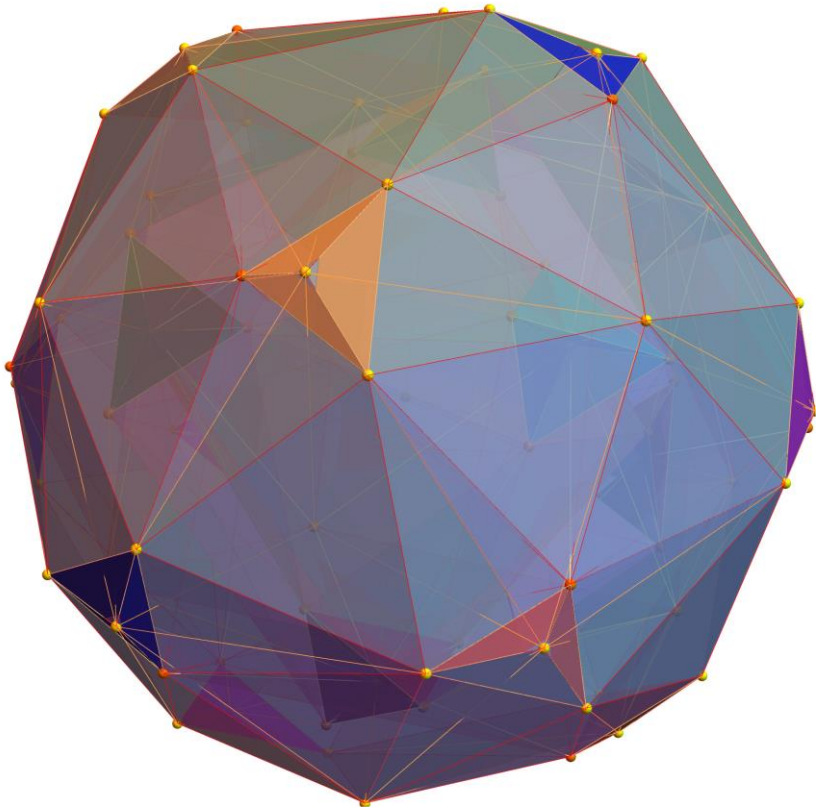


The 600-Cell / 120-Cell Quaternion 3D Structure

Dual Snub 24-Cell (144) and Alternate Dual Snub 24-Cell

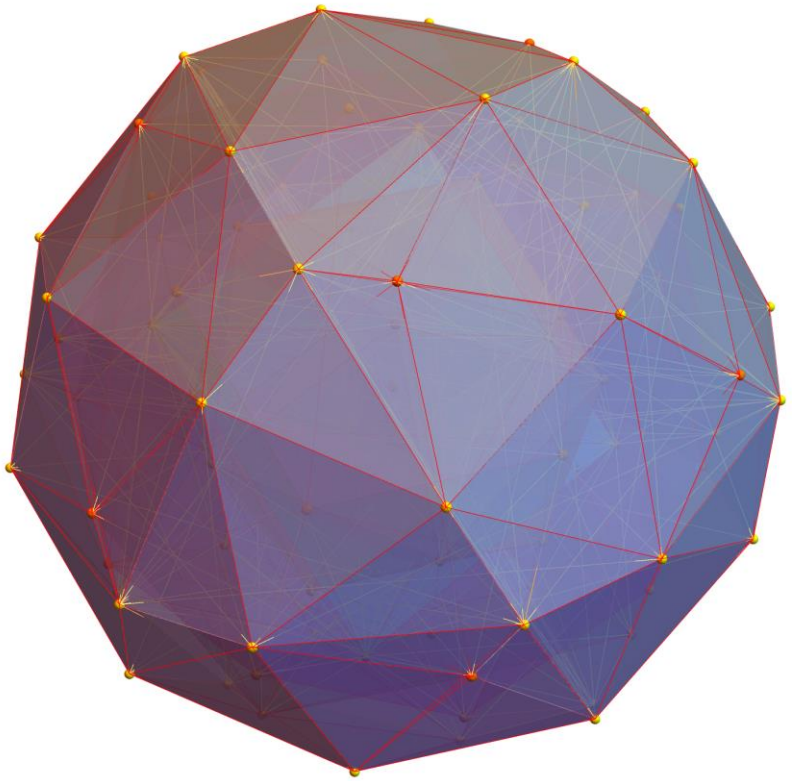
$T \oplus T' \oplus S'$

Out[ ]=



$T \oplus T' \oplus S$

Out[ ]=

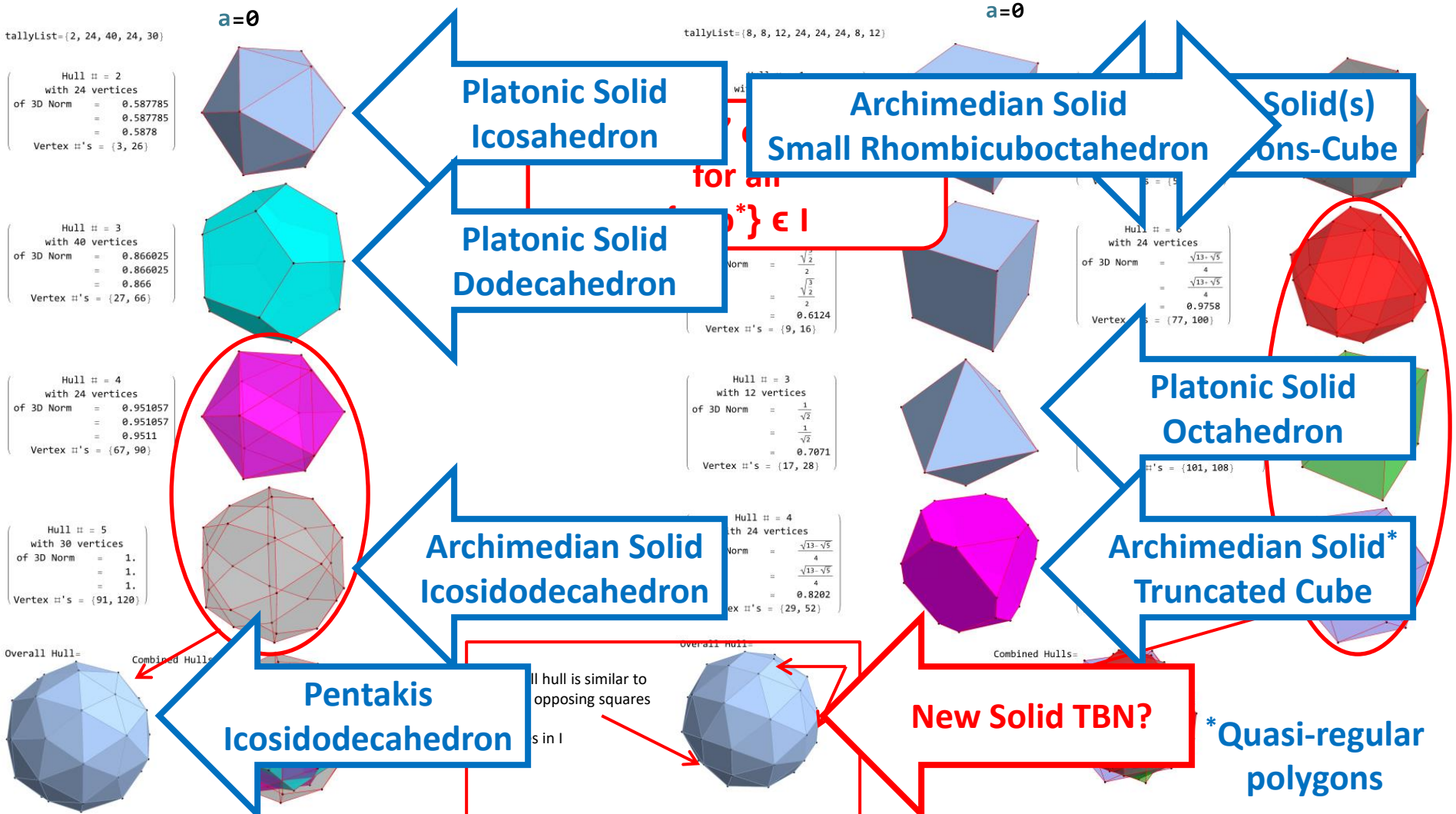


The 600-Cell / 120-Cell Quaternion 3D Structure

3D Hull Visualization of the H4 600-Cell Real & Quaternion Icosian (I)

I (120) and Alternate I' (120)

$$I = \sum_{a=0}^{4} \oplus p^a \circ T \qquad \alpha = p = \frac{1}{2}\left(\varphi + e_1 + \frac{e_2}{\varphi}\right) \qquad I' = \sum_{a=0}^{4} \oplus p^a \circ T'$$

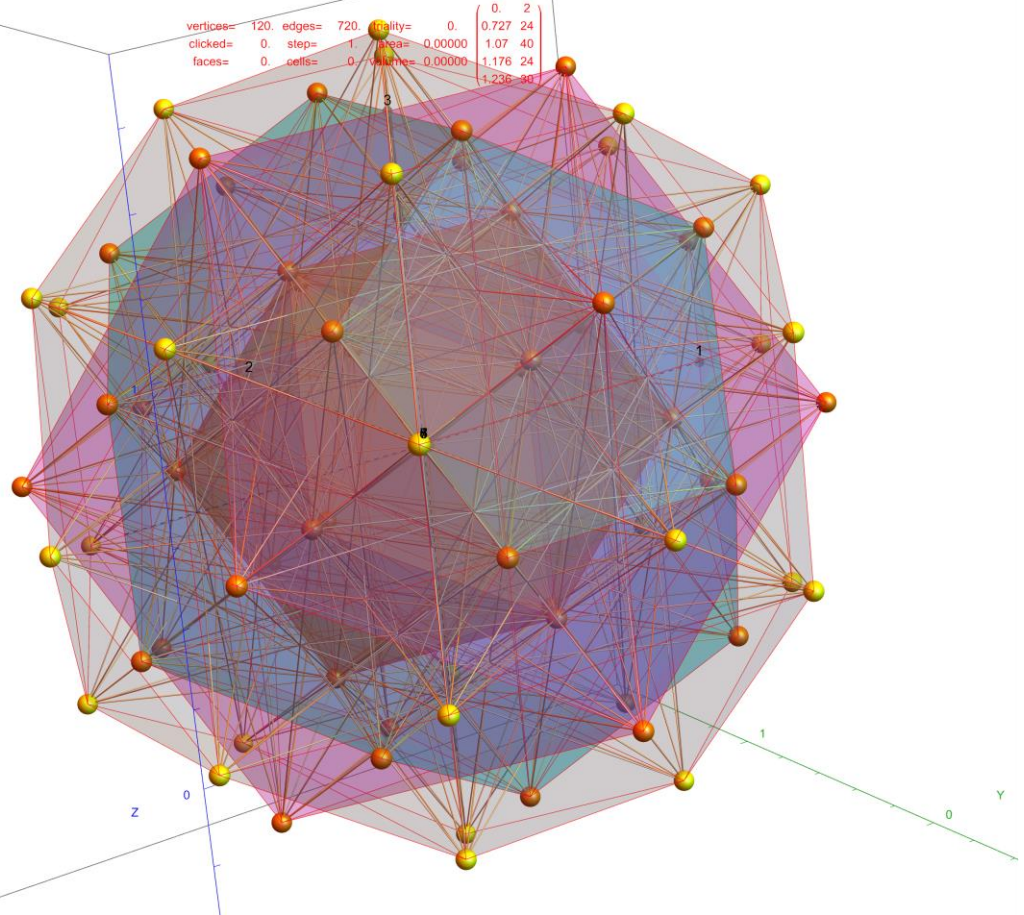


The 600-Cell / 120-Cell Quaternion 3D Structure

3D Hull Visualization of the 600-Cell Real & Quaternion Icosian (I)=H4

E8(SRE)↔ H4(120)+H4ϕ(120) with particle assignments

The 600-cell has an outer 3D hull of a Pentakis Icosidodecahedron (as shown above in slide #51)
Which is the dual to the Truncated Rhombic Triacontahedron or Chamfered Dodecahedron
(being the outer 3D hull of the 120-Cell orthogonally projected to 3D)



The 600-Cell / 120-Cell Quaternion 3D Structure

- It is important to note that all the indices below also work for both the canonical 120-Cell (J) values as well as an alternate form (J').
  - This (J) is implemented natively inside the VisibLie_E8 viewer based on the Koca's quaternion generation reproduces the canonical permutation vertex values exactly using $I = \sum_{a=0}^{4} \oplus p^{i} \circ T$, where $p=\alpha=\frac{1}{2}\left(\varphi + e_1 + \frac{e_2}{\varphi}\right)$.
    - This equality allows re-sorting of the canonical values to index to the I(120)x5 quaternion powers.
  - The (J') alternate T quaternion generated vertex set is implemented by using the auxData.xls input data file.
    - This file contains the same sort as the J for its vertex data exported from Mathematica, which allows the use of the same Weyl W(D4) orbit indices.

The 600-Cell / 120-Cell Quaternion 3D Structure

3D Hull Visualization of the 120-Cell (J) and Alternate J' (600) .

Using I and one exponent (a=0-4)

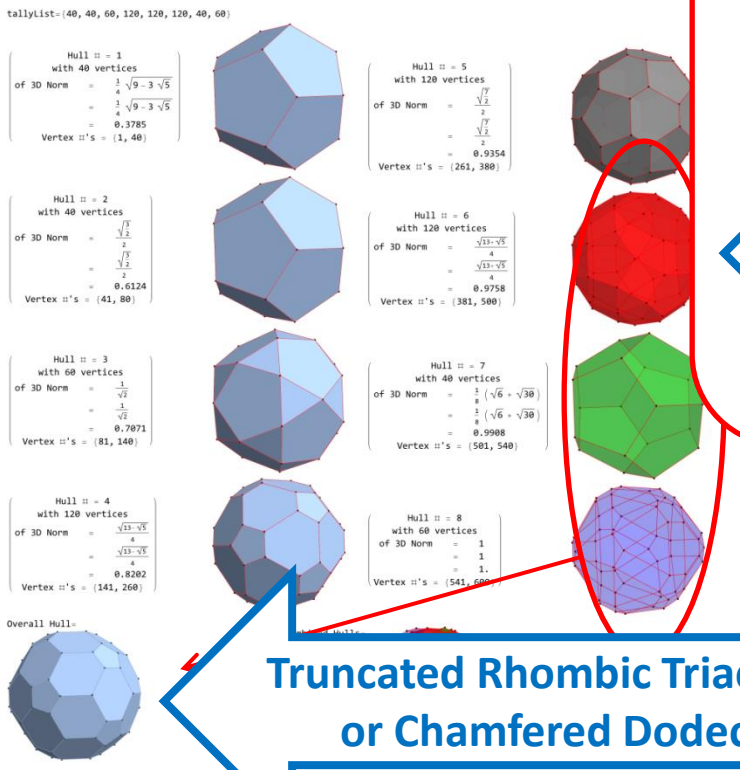
Orthogonal projection in 3 dimensions with

{x=1,y=1,z=1,w=0}

all other permutations simply rotate

$$c' = \frac{e_2 - e_3}{\sqrt{2}} \qquad \alpha = \frac{1}{2}\left(\varphi + e_1 + \frac{e_2}{\varphi}\right)$$

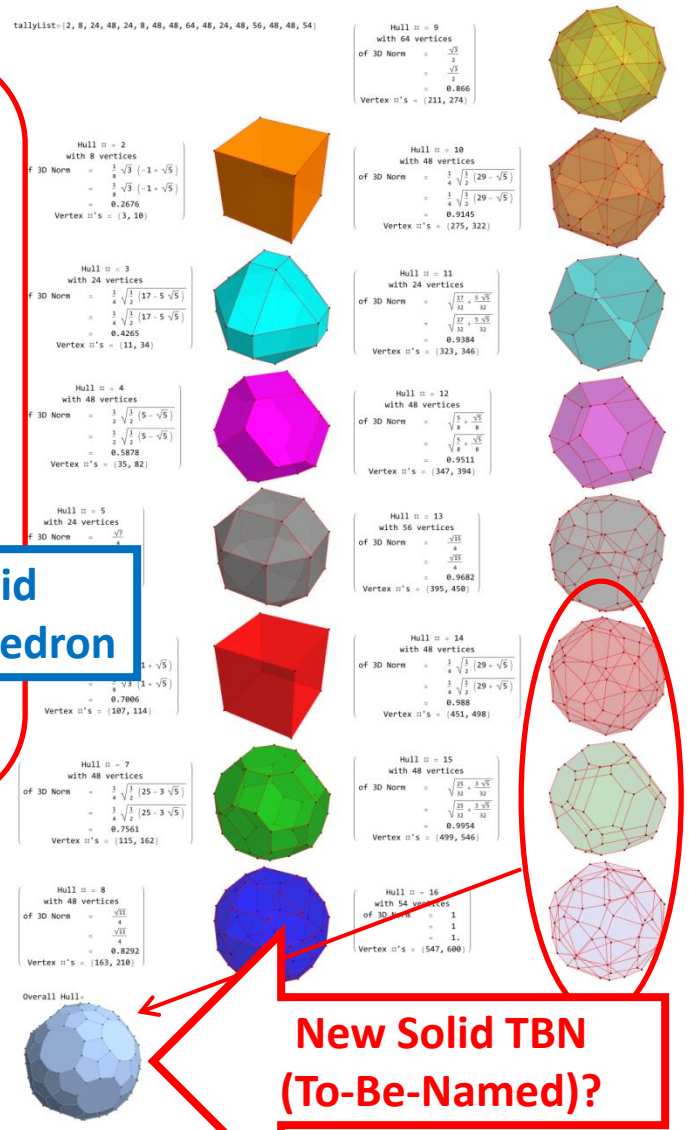
$$J = \sum_{a=0}^{4} \bigoplus I \circ \left(\alpha^a \circ \left(\overline{\alpha}^{\dagger a} \circ c'\right)\right)$$



~½ the permutations of α ϵ I w/c'(or c) ϵ T'(or T) produce these J, or J', (or 600 vertex hulls of I, I', T, or T') on each of the 4 3D subsets.

Wi other perms not le uniform faces geometry.

Archimedian Solid Rhombicosidodecahedron



Truncated Rhombic Triacontahedron or Chamfered Dodecahedron

New Solid TBN (To-Be-Named)?

The 600-Cell / 120-Cell Quaternion 3D Structure

3D Hull Visualization of the 120-Cell (J) and Alternate J' (600) .

Using two exponent (a=0-4,b=0,4)

$$\alpha = -\frac{1+\varphi\,(\varphi\, e_1 + e_3)}{2\,\varphi}$$

$$J = \sum_{a,b=0}^{4} \oplus \alpha^a \circ (T' \circ \beta^b)$$

$$\beta =$$

$$\alpha^a \circ (T \circ \beta^b)$$

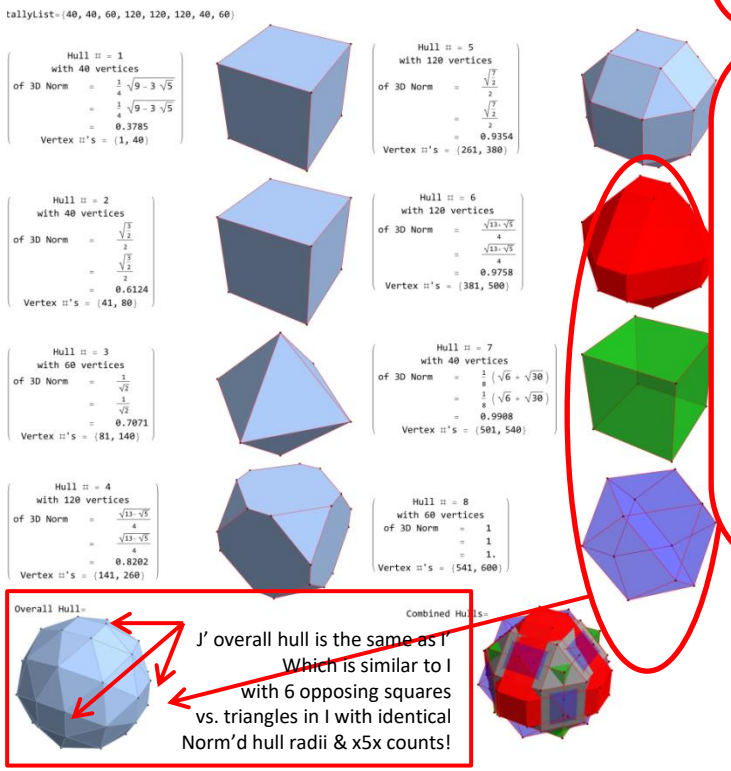
~½ the permutations of {α,β} as functions of c ϵ T & q ϵ I produce J, J', I, I', T, or T' on each of the 4 3D subsets.

New Solid TBN?

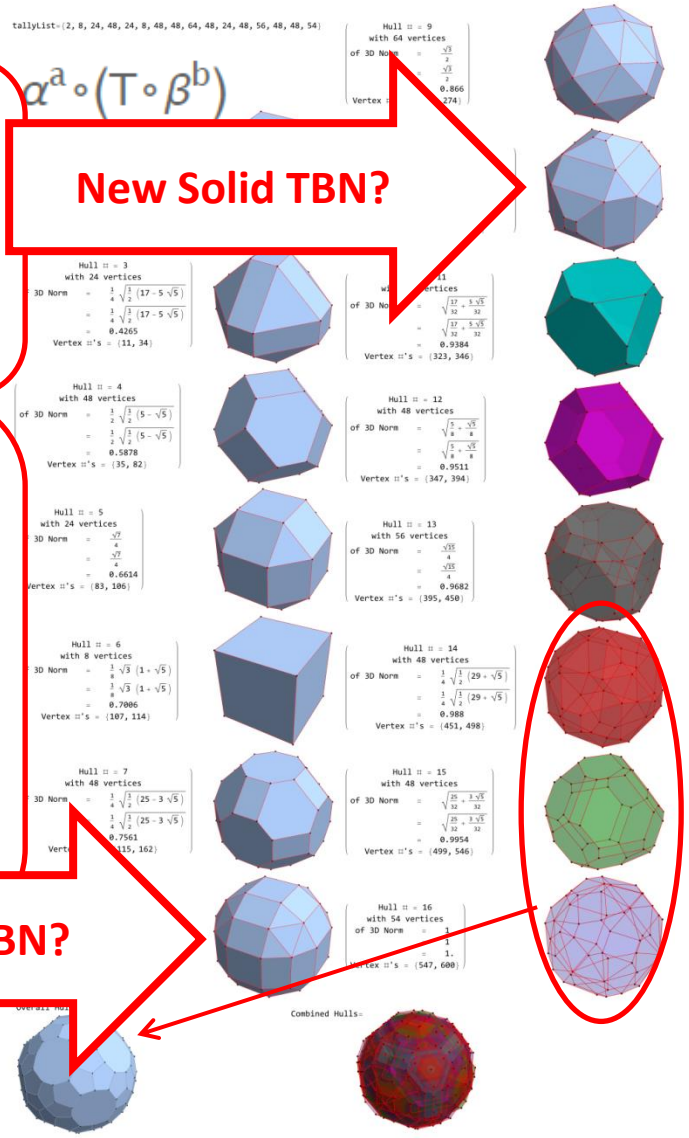
This J' is identical to the 1 exponent J' →

← This J has identical Norm'd hull radii and vertex counts to the 1 exponent J but with different hull

New Solid TBN?



J' overall hull is the same as I'
Which is similar to I
with 6 opposing squares
vs. triangles in I with identical
Norm'd hull radii & x5x counts!

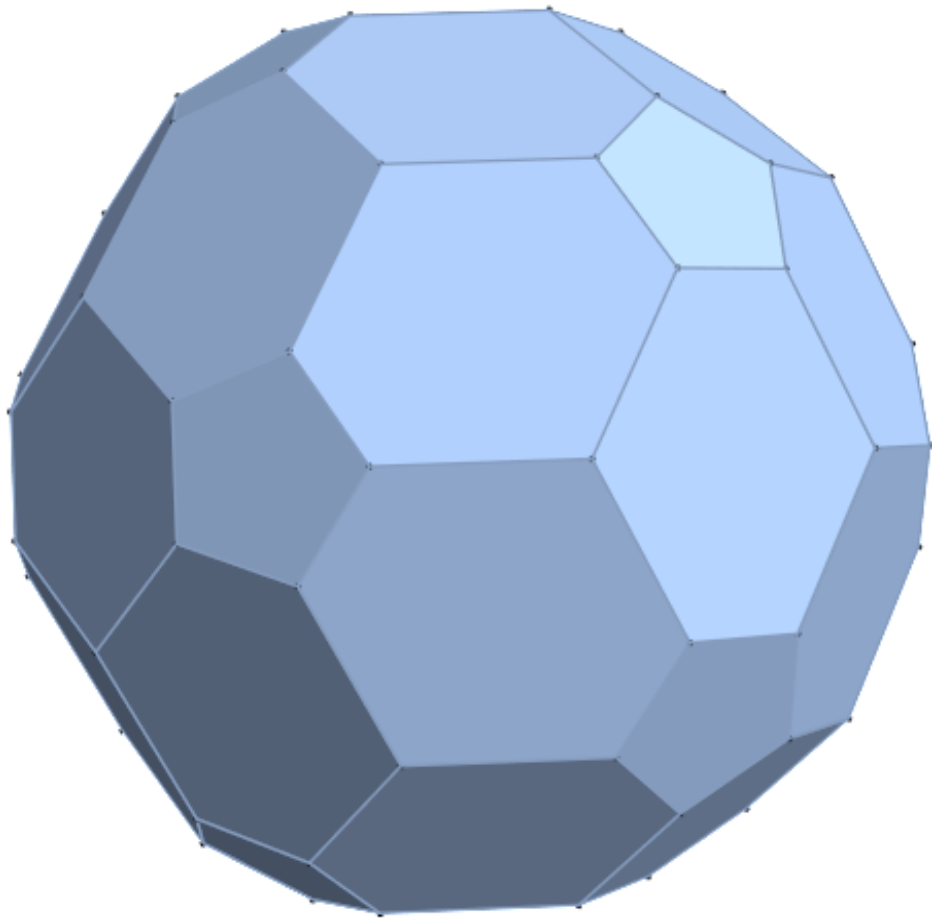


The 600-Cell / 120-Cell Quaternion 3D Structure

3D Visualization of the 120-Cell (J):

$$J = \sum_{a=0}^{4} \bigoplus I \circ \left(\alpha^{a} \circ \left(\overline{\alpha}^{\dagger a} \circ c'\right)\right)$$

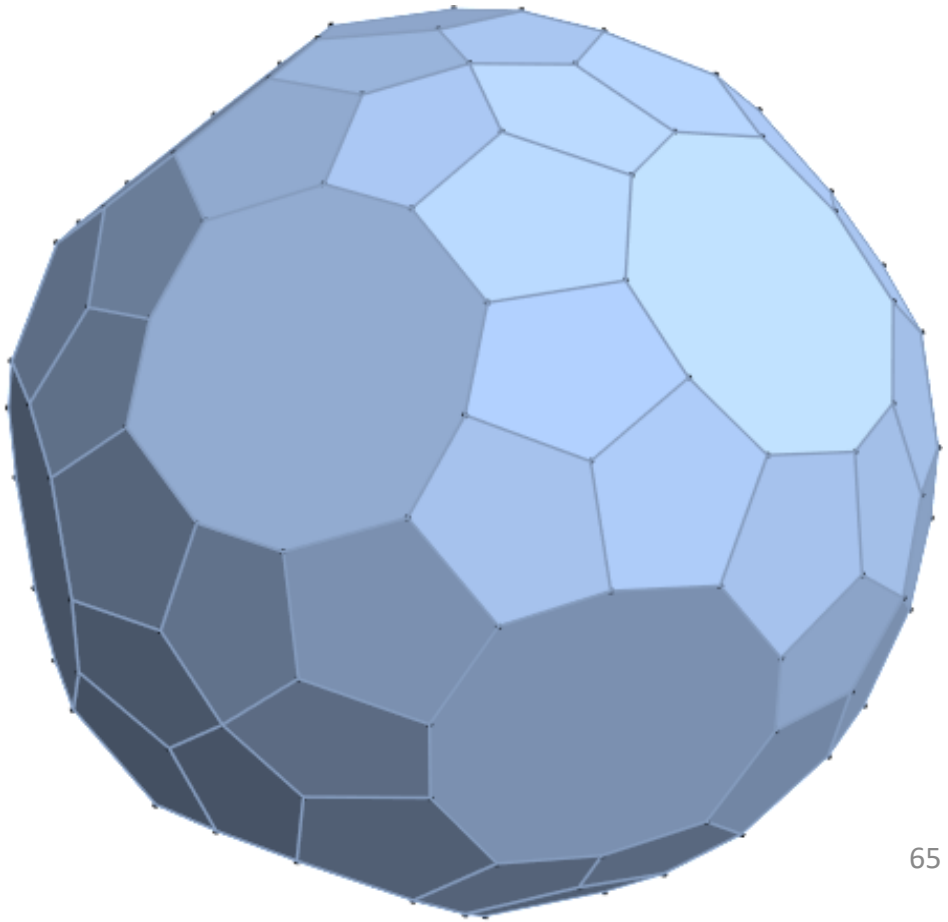
The outer 3D hull is the Truncated Rhombic Triacontahedron or Chamfered Dodecahedron



3D Visualization of the 120-Cell (J’) with particle assignments produced with:

$$J' = \sum_{a,b=0}^{4} \oplus \alpha^{a} \circ \left(T \circ \beta^{b}\right)$$

Is this outer 3D hull a new near-miss Johnson Solid?

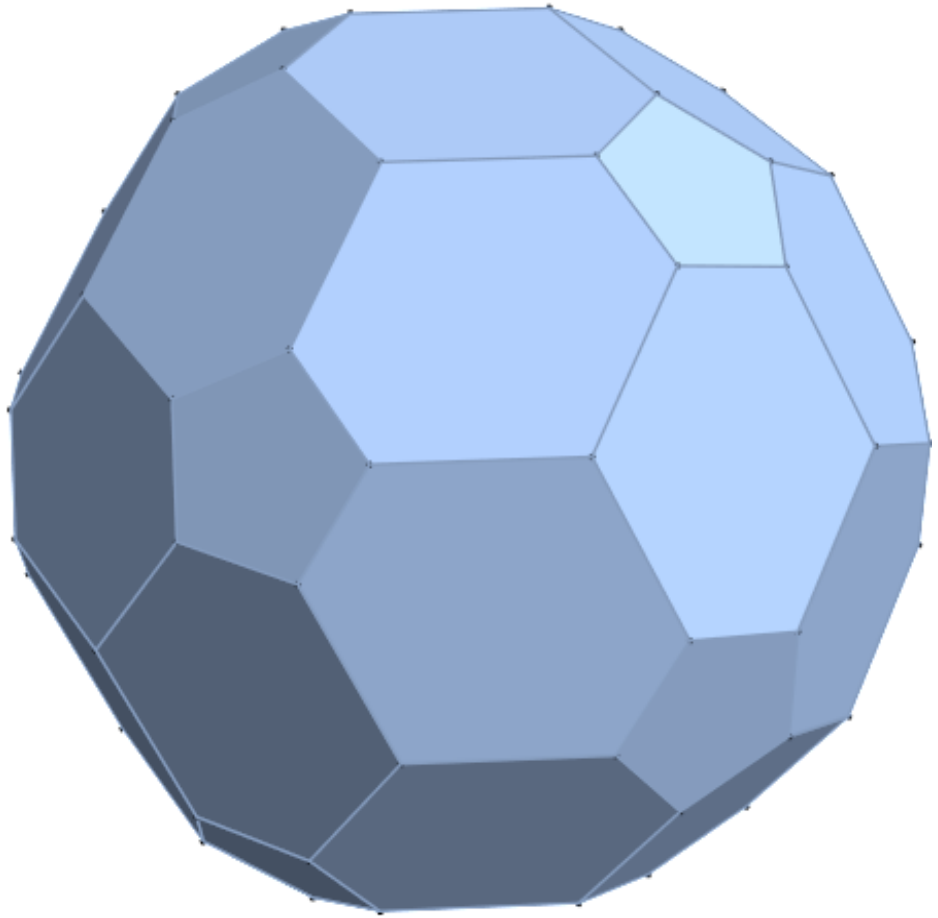


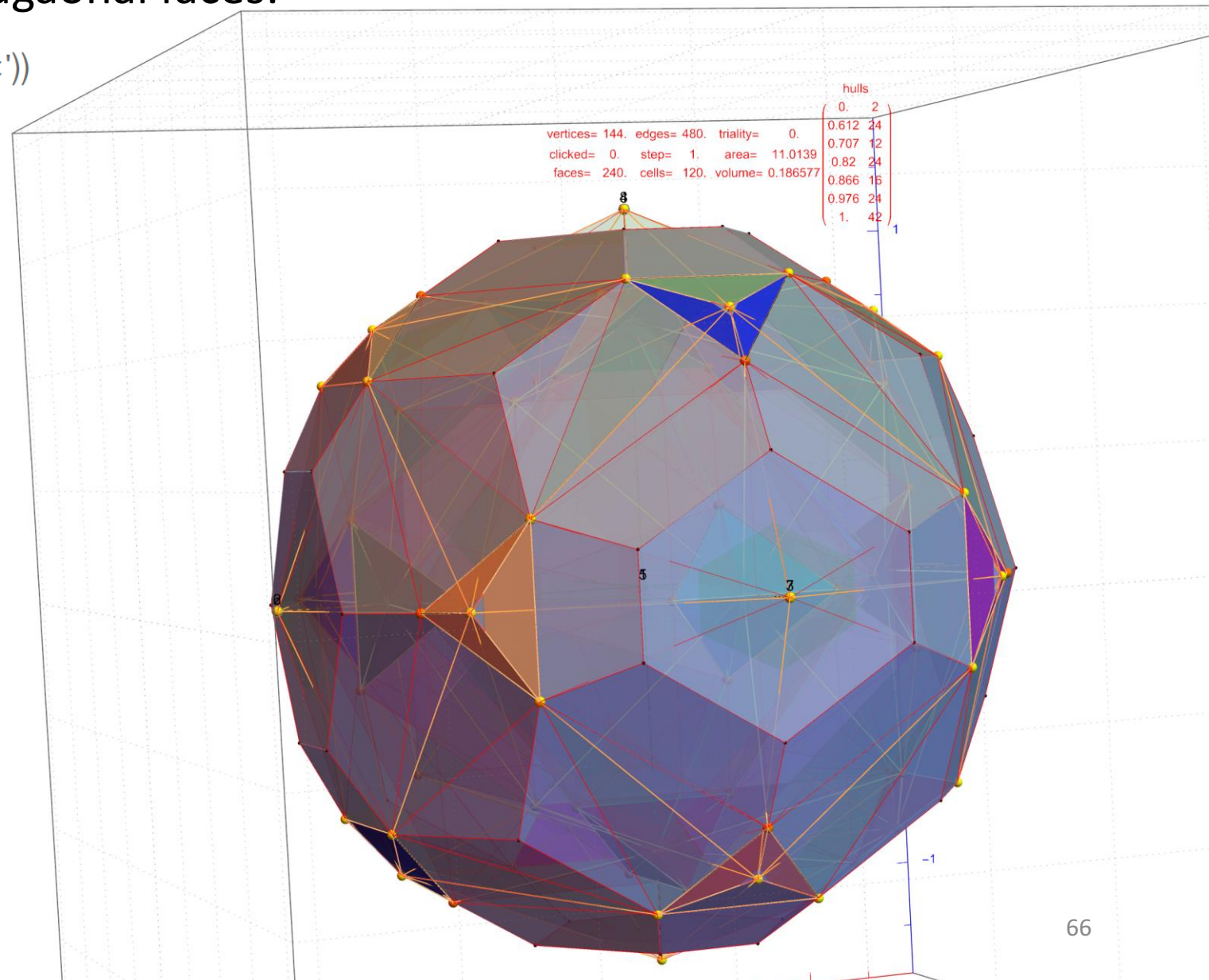
The 600-Cell / 120-Cell Quaternion 3D Structure

3D Visualization of the 120-Cell (J) with an overlay of the Dual Snub 24-Cell to the right showing the Tetrahedrons establishing the pentagaonal faces:

$$J = \sum_{a=0}^{4} \bigoplus I \circ \left(\alpha^{a} \circ \left(\overline{\alpha}^{\dagger a} \circ c'\right)\right)$$

The outer 3D hull is the Truncated Rhombic Triacontahedron or Chamfered Dodecahedron



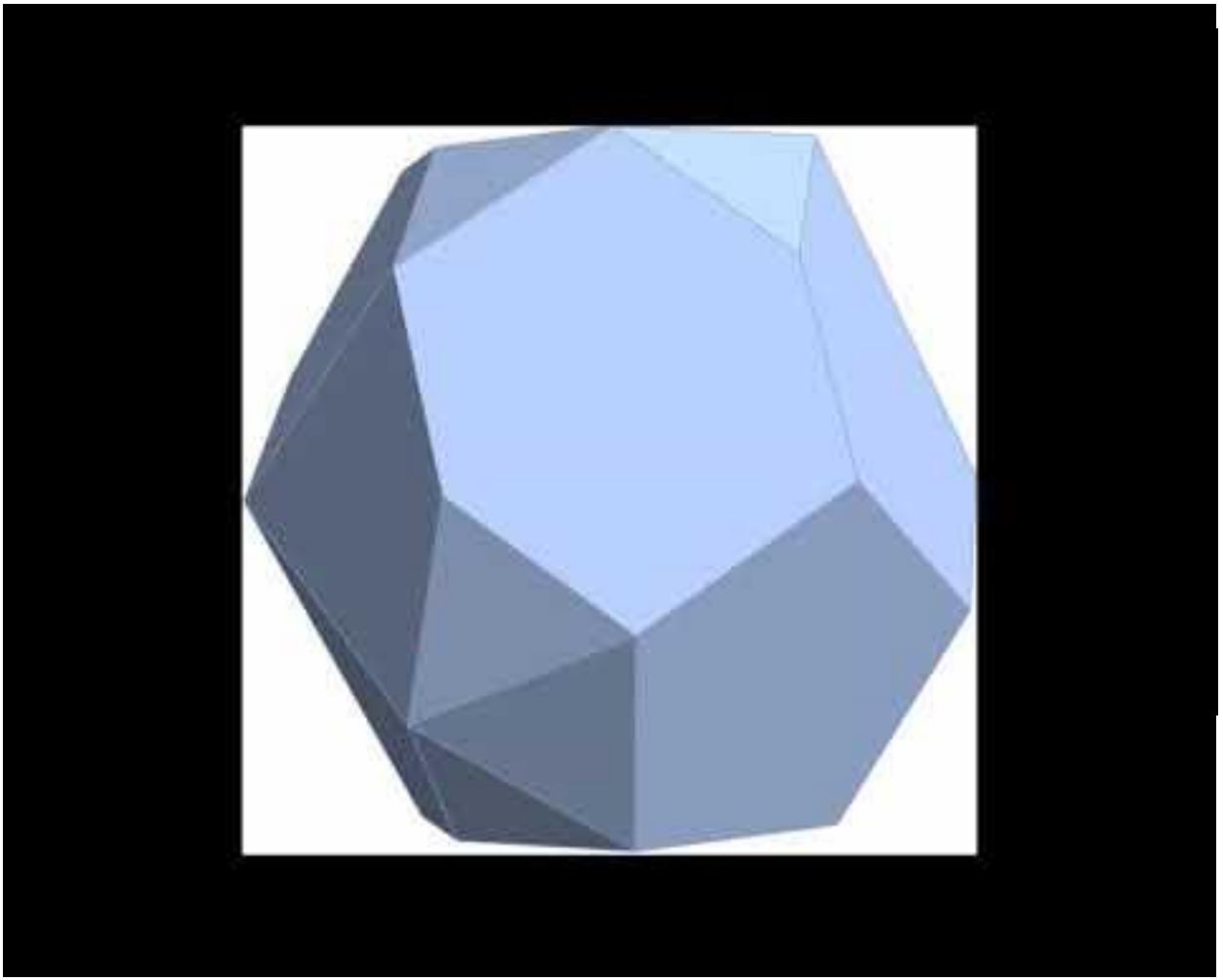


The 600-Cell / 120-Cell Quaternion 3D Structure

24-Cell (T) to 600-Cell (I) and 600-Cell (I) to 120-Cell (J)

$$I = \sum_{a=0}^{4} \oplus p^{i} \circ T$$

$$J = \sum_{a=0}^{4} \oplus I \circ (\alpha^{a} \circ (\overline{\alpha}^{\dagger a} \circ c'))$$



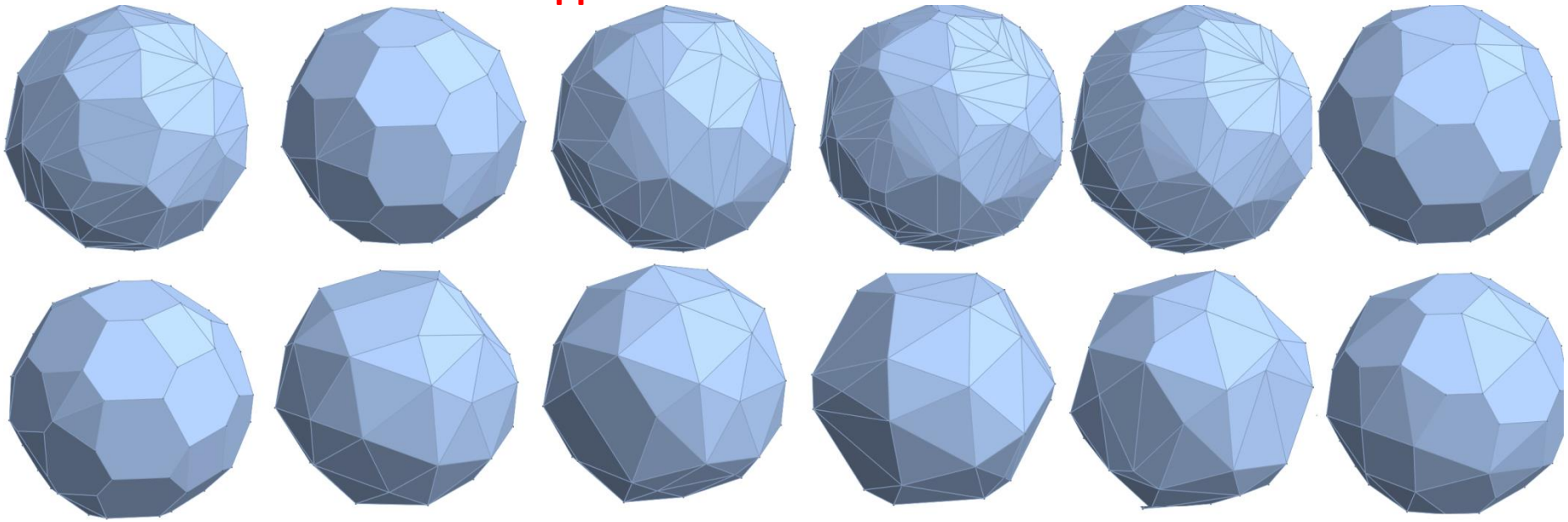
Animations building from exponent a=0-4

The 600-Cell / 120-Cell Quaternion 3D Structure

3D Hull Visualization of the 120-Cell (J) and Alternate J' (600) .

Examples of permutations that are not uniform within the 4 3D subsets of a 4D polytope and having less uniform faces geometry.

Some approximate the Dual Snub24-Cell.



These are natural rotations of a 4D polytope when projected to 3D. The fact they emerge within the limited quaternion exponents is interesting.